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Gamma functions and Gauss sums for function fields and periods of Drinfeld modules

Thakur, Dinesh Shraddhanand, Ph.D.

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Typed name Barry Mazur

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Typed name Benedict Gross

Date May 15, 1987

Gamma functions and Gauss sums for function fields and periods of Drinfeld modules

A thesis presented

by

Dinesh Shraddhanand Thakur

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
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Abstract

We define gamma functions for function fields and interpolate them at all places, refining the construction due to Carlitz and Goss for the rational function field by including the degree part and generalising it. We prove reflection and multiplication formulae for them and relate the special values, for the infinite place, to the periods of appropriate Drinfeld modules. At the finite places, the special values are related to Gauss sums (taking values in function fields), that we define, so as to give an analogue of the Gross-Koblitz theorem. We also prove analogues of many classical and recent results such as Stickelberger's theorem, the Hasse-Davenport theorem, Weil's theorem on Jacobi sums as Hecke characters and Deligne's theorem. We also prove an analogue of the Chowla-Selberg formula, in a very special case. These theorems lead also to transcendence and algebraicity results for the special values of gamma functions. Results about the transcendence of the special values of the Carlitz zeta function at 'odd' positive integers and their ratio with the appropriate powers of the period of the Carlitz module are also obtained. In particular, we prove that the value of the Carlitz zeta function at infinitely many 'odd' positive integers is transcendental.

Contents

Introduction

Various analogies between the theory of number fields and the theory of function fields have been discovered and exploited by many mathematicians to test and guess conjectures and to get better understanding of the issues. Good examples are Iwasawa theory, Arakelov theory; conjectures of Birch and Swinnerton-Dyer, Mordell, Stark, Shimura-Taniyama-Weil, Riemann hypothesis and so on.

Another chapter of such analogies was opened by the work of Carlitz and Drinfeld. These had to do with explicit class field theory, Langlands conjectures, complex multiplication etc. These themes were further developed by Goss, Hayes, Gekeler, Yu etc.

The results in this thesis belong to this area. Carlitz and Goss had defined gamma functions for the ring of polynomials over a finite field. We generalise the construction (this generalisation being different than the one suggested by Goss) to an arbitrary ring of integers (or an order), or an ideal in it, and using formulae of Carlitz and Hayes-Gekeler, relate the special values to the periods of appropriate Drinfeld modules. Following a suggestion by E.U. Gekeler, we take care of the degree of the period too. Reducing the reflection and multiplication formulae to the computation of the digit expansion of p-adic numbers, we prove them in a general context.

We also prove an analogue of Chowla-Selberg formula in a very special case.

Giving a new interpretation to the notion of additive character, we define Gauss sums as 'character' sums, taking values in function fields and prove analogues of many classical and recent results, such as the Stickelbeger theorem, the Hasse-Davenport theorem, Weil's theorem on 'Jacobi sums as Hecke characters', Gross-Koblitz and Deligne's results concerning the special values of gamma functions and Gauss sums. We do not need here to use an analogue of Katz' computation involving p-adic cohomology, which was a crucial step in the Gross-Koblitz argument. It may be possible to give another proof involving such a computation.

Apart from the transcendence results that we obtain for the special values of gamma function, we show that the special values of Carlitz' zeta function at infinitely many 'odd' positive integers are transcendental, by using results of Wade. We also have some results about the ratios of these values with the appropriate powers of the periods of the Carlitz module. For more details, please refer to the introductions to the individual chapters. Chapter four can be read independently of chapters two and three.

Some of the results, especially in the chapter three, are not presented in the generality which they could have been, because of the time constraints. After communicating these results to David Hayes, I was informed by him that the result about Weil's theorem on the Hecke characters had been done in much more generality by him (unpublished manuscript) and the notion was first introduced by Gross [Gr 1]. We do not include the connection with L-functions and Stark's conjectures. (See [Ha 1]).

Remark on the notation: It should be noted that the symbols ζ, Π, Γ, D_i, $\pi, \tilde{\pi}, \tilde{D}_i$ etc. signify different objects according to the context. We follow the standard convention where the empty sum evaluates to zero and the empty product evaluates to one. A prime over the summation or the product symbol means that the variable runs only through nonzero values. Also, expansion in powers of q, usually denotes the standard digit expansion.

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This thesis is dedicated to the memory of Vasantrao Deshpande.

Chapter 1

Background

This chapter sets up the basic framework and describes the work of Carlitz, Drinfeld and Goss, which is relevent to this thesis. More precisely, we introduce Carlitz's zeta and gamma functions for $\mathbf{F}_q[T]$ and their interpolations due to Goss and explain the basic set-up of Drinfeld modules.

1.1 The Carlitz zeta function

The Riemann zeta function is given by

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad s \in \mathbf{C} \quad \mathrm{Re}(s) > 1$$

By interpreting 'n' as the cardinality of the residue class ring (i.e. the norm) of a nonzero ideal (n) in $\mathbf{Z}$ and carrying over the interpretation to function fields, Artin developed zeta functions of a complex variable for function fields and showed that they gave an excellent analogue.

Carlitz thought of 'n' as a positive or monic (there are two choices ± 1 of signs for $\mathbf{Z}$ whereas there are $q-1$ choices in $\mathbf{F}_q^\times$ for $\mathbf{F}_q[T]$), representative of a nonzero ideal (n). Hence he defined [Ca 1] a zeta function for $\mathbf{F}_q[T]$ by

$$\zeta(s) = \sum_{n \text{ monic in } \mathbf{F}_q[T]} 1/n^s \quad s \in \mathbf{N}, \zeta(s) \in \mathbf{F}_q((1/T))$$

In other words, instead of the 'norm' which just depends on the degree of the polynomial, he used the whole polynomial, trading it for a smaller domain for s. The justification lies in the following theorem[Ca 1].

Theorem 1.1 *If $\zeta(s)$ denotes the Carlitz zeta function for $\mathbf{F}_q[T]$, defined above, then*

$$\zeta((q-1)m) = \frac{B_{(q-1)m}}{\Pi((q-1)m)}\tilde{\pi}^{(q-1)m}$$

where m is a positive integer, $B_{(q-1)m} \in \mathbf{F}_q(T)$, $\Pi((q-1)m) \in \mathbf{F}_q[T]$ and $\tilde{\pi}$ is a transcendental number over $\mathbf{F}_q(T)$.

Example:

$$\zeta(q-1) = \frac{\tilde{\pi}^{q-1}}{T^q - T}$$

This theorem should be compared with the corresponding result for the Riemann zeta function, namely

$$\zeta(2m) = \frac{B_{2m}}{2(2m)!}(2\pi)^{2m}$$

where $B_{2m} \in \mathbf{Q}$, $(2m)! \in \mathbf{Z}$, 2π is transcendental over $\mathbf{Q}$. Hence one should think of multiples of $(q-1)m$ as analogues of 'even integers', $B_{(q-1)m}$ as analogues of Bernoulli numbers B_{2m} (Carlitz proved von-Staudt type congruences for them to justify the terminology) and $\Pi((q-1)m)$ as factorials and $\tilde{\pi}$ as an analogue of $2\pi i$ (these analogies and idea of the proof will be explained later).

David Goss [Go 1] showed how just grouping together the terms of the same degree gives an interpolation of the zeta function from its original domain of natural numbers to a much bigger domain containing in particular the integers, which is all that we need for our present purposes.

Theorem 1.2

$$\zeta(s) = \sum_{i=0}^{\infty} \sum_{\substack{n \in \mathbf{F}_q[T] \\ n \text{ monic} \\ \deg(n)=i}} n^{-s}$$

is a well-defined element of $\mathbf{F}_q((1/T))$ for $s \in \mathbf{Z}$. Suppose $s < 0$. Then

$$\zeta(s) \in \mathbf{F}_q[T],\ \zeta(s) = 0 \textit{ iff } s \equiv 0 \bmod (q-1)$$

This theorem should be compared with the corresponding result for the Riemann zeta function, namely that its values at negative integers are rational, and zero or not according to whether the argument is even or odd. The proof follows by writing a monic polynomial n of degree i as $Th + b$ with h of degree $i - 1$ and $b \in \mathbf{F}_q$, and using the binomial theorem to get the induction formula

$$\zeta(-i) = 1 - \sum_{\substack{f=0 \\ (q-1)|(i-f)}}^{i-1} \binom{i}{f} T^f \zeta(-f)$$

which shows $\zeta(-i) \in \mathbf{F}_q[T]$ since $\zeta(0) = 1$. If $(q-1)$ divides i, then induction shows $\zeta(-i) = 1 - 1 + 0 = 0$. If $(q-1)$ does not divide i, then there being no term in the summation corresponding to $f = 0$, $\zeta(-i) = 1 - Tp(T)$ which can not be zero ($p(T)$ is just some polynomial here).

Goss [Go 1] further observed that, essentially since $\zeta(-i)$ is a finite sum over n's of n^i and since for a prime P of $\mathbf{F}_q[T]$ of degree d and n relatively prime to P, n^i as a function of i interpolates to a continuous function on $S = \mathbf{Z}/(q^d - 1) \times \mathbf{Z}_p$ (For, if $i \equiv 0 \bmod (q^d - 1)p^j$ then $n^i \equiv 1 \bmod P^j$), by throwing out the Euler factor at P, we get P- adic zeta function with values in $\mathbf{F}_q[T]_P$

$$\zeta_P(-i) = \sum_{i=0}^{\infty} \sum_{\substack{(n,P)=1 \\ \deg n = i \\ n \text{ monic}}} n^i \quad i \in S$$

Observe that if $q = 2$, we get identically zero function, because $\zeta(-i) = 0$ and $\zeta_P(-i) = (1 - P^i)\zeta(-i)$ for $i \in \mathbf{N}$. In general, it is zero on the closure of $-(q-1)\mathbf{N}$ in S. Hence Goss [Go3] redefines the zeta function on $-(q-1)\mathbf{N}$ by replacing the values of zeta by values of partial derivatives of zeta function on the bigger domain. This amounts to putting $\zeta(0) = 0$ and using the same induction formula given above for $\zeta(-i)$. After removing the Euler factor, it again extends to a continuous function on S.

1.2 Carlitz' factorial function

Carlitz' factorial function Π for $\mathbf{F}_q[T]$, which appeared in theorem 1.1 was defined by him as follows.

Define $[i], D_i$ for non-negative integers i by,

$$[0] = D_0 = 1,\ [i] = T^{q^i} - T,\ D_i = [i]D_{i-1}^q,\ i \in \mathbf{N}$$

For $z \in \mathbf{N}$, $z = \sum u_i q^i$, $0 \le u_i \le q-1$, define the factorial function by

$$\Pi(z) = \prod D_i^{u_i} \in \mathbf{F}_q[T]$$

In addition to its analogous occurence in the special values of zeta function (see theorem 1.1), its prime factorisation

$$\Pi(n) = \prod_{P \text{ monic prime}} P^{n_P},\ n_P = \sum_{e=1}^{\infty} [n/NP^e]$$

(where (the norm) NP is the cardinality of the residue class field) is an exact analogue of the classical formula

$$n! = \prod_{p \text{ positive prime in } \mathbf{Z}} p^{n_p},\ n_p = \sum [n/Np^e]$$

This was noticed by W.Sinnott and is easy to prove from the definitions since $[i]$ is the product of all monic primes whose degree divides i and D_i is the product of all monic polynomials of degree i.

Remark: This formula does not hold in general, for the definition of gamma functions we propose in the next chapter, or for the definition proposed by Goss. Also, Goss in his papers called, what we call factorial function, gamma. We put $\Gamma(z) = \Pi(z-1)$ in accordance with the classical convention, for all 'factorials'.

Goss [Go 2, appendix] made v-adic interpolations of the factorial at all places v of $\mathbf{F}_q[T]$ as follows. Since $D_i = T^{iq^i} - T^{(i-1)q^i + q^{i-1}} +$ lower degree terms, the unit part

$$\overline{D_i} = D_i / T^{\deg D_i} = 1 - 1/T^{(q-1)q^{i-1}} + \cdots \to 1,\ i \to \infty$$

in $\mathbf{F}_q((1/T))$. So the unit part of $\Pi(n)$ interpolates to a continuous function (∞-adic factorial) $\overline{\Pi}(n)$;

$$\overline{\Pi} : \mathbf{Z}_p \to \mathbf{F}_q((1/T)),\ \sum u_i q^i \to \prod \overline{D_i}^{u_i}$$

Also, since D_i is the product of all monic elements of degree i, we have a Morita- style v-adic factorial $\Pi_v : \mathbf{Z}_p \rightarrow \mathbf{F}_q[T]_v$ for finite primes v of $\mathbf{F}_q[T]$ given by

$$\Pi_v(z) = \prod(-D_{i,v})^{u_i}$$

where $D_{i,v}$ is the product of all monic elements of degree i, which are relatively prime to v. This makes sense since $-D_{i,v} \rightarrow 1$ v-adically as $i \rightarrow \infty$.

1.3 Drinfeld modules

Let K be a function field of one variable with field of constants $k = \mathbf{F}_q$ (where q is a power of a prime p),let ∞ be a place of K, δ be the residue class degree of ∞, A be the ring of integers outside ∞ of K, K_∞ be the completion of K at ∞, k_∞ be the field of constants $\mathbf{F}_q^\delta$ of K_∞ and Ω be the completion of an algebraic closure of K_∞.

Drinfeld [Dr 1] introduced the key concept of an elliptic module (also called Drinfeld module) in the theory of function fields, and successfully attacked the Langlands conjectures for function fields.

A Drinfeld module is an analogue of an elliptic curve (more so for rank one and two), or of an elliptic curve with complex multiplication.To motivate the definition, notice that classically an elliptic curve is just $\mathbf{C}/\Lambda$ where lattice Λ is just a rank two $\mathbf{Z}$-submodule of $\mathbf{C}$. For a finite extention L of K_∞, a lattice Λ over L will be, by definition , a finitely generated discrete A-submodule in L^{sep} invariant with respect to $\text{Gal}(L^{\text{sep}}/L)$. An Important point is that, in contrast to the classical case, lattices of arbitrary rank exist.

Given such a lattice Λ, the corresponding exponential function (like exponential, sine or Weierstrass σ-function)

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}'(1 - z/\lambda)$$

is an entire additive function and induces a group isomorphism $\bar{L}/\Lambda \equiv \bar{L}$. Since Λ is an A module, we get $\bar{L}$ as an A module. In fact, for $a \in A, a \neq 0$, by comparing divisors, we see that

$$e_\Lambda(az) = c \prod_{b \in \frac{1}{a}\Lambda/\Lambda} (e_\Lambda(z) - e_\Lambda(b)) = \rho_a(e_\Lambda(z))$$

where ρ_a is a polynomial function and $c \in \Omega$.

So from the analytic object $\overline{K_\infty}/\Lambda$, we extracted a nice algebraic object ρ. So we have an A-module structure on the additive group scheme $\mathbf{G_a}$ i.e. a non trivial embedding

$$\rho : A \rightarrow \mathrm{End}_L \mathbf{G_a} = L\{F\}$$

where $L\{F\}$ is a twisted polynomial ring in Frobenius i.e. $\mathbf{F}_p$-algebra generated by elements of L and by F with the commutation relation $Fl = l^p F$. This leads to

Definition: Let L be a field over A i.e. $i : A \rightarrow L$ be a homomorphism. Elliptic (or Drinfeld) A-module over L is a homomorphism $\rho : A \rightarrow L\{F\}, (a \rightarrow \rho_a)$ such that the constant term of ρ_a is just $i(a)$ and ρ is not just $i : A \rightarrow L \subset L\{F\}$ (non-triviality condition).

This is analogous to 'elliptic curve over L with complex multiplication by A'. One should keep in mind the classical situations $\mathbf{Z} \rightarrow \mathrm{End}G_m, \mathbf{Z} \rightarrow \mathrm{End}E$,where E is an elliptic curve, and $\mathcal{O}_F \rightarrow \mathrm{End}E$ (where F is an imaginary quadratic field and E is a complex multiplication elliptic curve).

Definition : Let ρ, ρ' be Drinfeld A-modules over L.
(1)Isogeny over L from ρ to ρ' is a nonzero element $\mu \in L\{F\}$ such that $\mu\rho_a = \rho'_a\mu$ for $a \in A$.
(2)If, further, $\mu \in L^\times$ then we say μ is an isomorphism over L.
(3) If $i : A \rightarrow L$ is an embedding, then ρ is said to be of generic characteristic.
(4) If the degree of ρ_a (viewed as polynomial in F) is $-r(\deg a)(\log_p q)$ for all $a \in A$, then we say that ρ has rank r.

It can be shown that, for any Drinfeld module ρ such r exists and is a natural number. Isogenous Drinfeld modules have the same rank.

From now on by 'Drinfeld module' we will always mean a Drinfeld module of generic characteristic.

Theorem 1.3 *If L is a finite extention of K_∞, then the category of elliptic A-modules of rank r over L is isomorphic to the category of A-lattices of rank r over L.(So, in particular, there exist elliptic A- modules of arbitrary rank over K_∞^{sep})*

We have already described how to pass from Λ to $\rho = \rho^\Lambda$ via $e_\Lambda = e_\rho$. On the other hand, given ρ, we can essentially solve for e_ρ via $e_\rho(az) = \rho_a(e_\rho(z))$ and then we can recover Λ as Ker e_ρ. (For more details see [Dr 1]).

Some forty years before Drinfeld defined the general notion, by different considerations, Carlitz [Ca 1] came across 'the simplest' Drinfeld module of rank one (called Carlitz module), namely the $\mathbf{F}_q[T]$-module over $\mathbf{F}_q(t)$ given by $T \to \rho_T = T - F$ (T being the generator of $\mathbf{F}_q[T]$, ρ is well defined as $\mathbf{F}_q$-algebra homomorphism. Also F is the 'q-th power Frobenius'(abuse of notation)). He showed that the corresponding exponential is given by

$$e_\rho(z) = \sum_{h=0}^{\infty} (-1)^h z^{q^h} / D_h$$

and the corresponding lattice Λ is $\tilde{\pi}\mathbf{F}_q[T]$ with D_h and $\tilde{\pi}$ as in the section 2 and 1 respectively. (This defines $\tilde{\pi}$ up to multiplication by $\mathbf{F}_q^\times$. It is enough for the purposes of theorem 1.1). He also showed

$$\tilde{\pi} = \lim [1]^{q^k/(q-1)} / [k] \cdots [1]$$

We will derive generalisations of these formulae in chapter 2.

Outline of the proof of the theorem 1.1 : We know that

$$e_\rho(z) = z \prod_{\lambda \in \Lambda}{}' (1 - z/\lambda)$$

so taking the logarithmic derivative we have

$$e_\rho'(z)/e_\rho(z) = 1/e_\rho(z) = \sum_{\lambda \in \Lambda} 1/(z + \lambda)$$

So

$$z/e_\rho(z) = 1 + \sum_{\lambda \in \Lambda}{}' z/\lambda - \sum_{\lambda \in \Lambda}{}' z^2/\lambda^2 + \cdots = 1 + \sum_{(q-1)|m} (\zeta(m)/\tilde{\pi}^m) z^m$$

since $(-1)^{(q-1)r} = 1$ in characteristic p and $\sum_{a \in \mathbf{F}_q^\times} a^r = -1$ or 0 according as $(q-1)|r$ or not. Now $e_\rho(z)$ has rational coefficients, so $\zeta(m)/\tilde{\pi}^m$ is rational if $(q-1)|m$. Making the calculations explicit [Ca 1], you get theorem 1.1.(See also chapter 4).

We end this chapter by briefly mentioning another beautiful application made by Carlitz of this theory. He showed that adjoining the roots in Ω of 'the cyclotomic equation' $\rho_a(z) = 0, a \in A = \mathbf{F}_q[T]$(i.e. the a-division points of the action by ρ) to $K = \mathbf{F}_q(T)$, we get an abelian extension with Galois group $(A/aA)^\times$. (This should be compared with the cyclotomic extension of $\mathbf{Q}$ obtained by adjoining the division points of $\mathbf{Z} \to \mathrm{End}G_m$ and the situation in Lubin-Tate theory). He developed an analogous cyclotomic theory for $\mathbf{F}_q[T]$. This aspect has been greatly generalised by the work of Drinfeld and Hayes.

Chapter 2

Gamma function and periods

In this chapter we define gamma functions in the quite general situation of section 1.3, prove their interpolations at finite primes, and compute some special values of interest. More precisely, the first section establishes a relationship between the special value of the Carlitz-Goss gamma function for $\mathbf{F}_q[T]$ at 0 and the period of the Carlitz module. In section two the multiplication and reflection type formulae for the gamma function are proved, the proofs applying in a much wider context. Then, in section three, we propose a definition for the gamma function of A, which also takes care of the valuation at ∞ in addition to the unit part, and generalise the results of section one. In the last section, we interpolate the gamma function at the finite primes and evaluate some of their special values, getting algebraicity results: analogues of Gross-Koblitz algebraicity results in a very special case. We obtain full Gross-Koblitz results in the next chapter.

2.1 The case $A = \mathbf{F}_q[T]$

Even though the results of section three will contain the results of this section as a special case and will even take care of the 'degree' of the period, we nonetheless include this section, because we already have the necessary machinery for this case, calculations are much more explicit and the complications in the general situation are absent.

In this section, for $M \in \mathbf{F}_q((1/T))$, $M/T^{\deg M}$ will be denoted by $\overline{M}$.

Recall the Carlitz' formula for the period $\tilde{\pi}$ given in section 1.3. Namely,

$$\tilde{\pi} = \lim [1]^{q^k/(q-1)}/[1]\cdots[k]$$

so $\tilde{\pi}^{q-1} \in \mathbf{F}_q((1/T))$ and $\overline{\tilde{\pi}^{q-1}}$ makes sense. By $\overline{\tilde{\pi}}$ we will denote its unique $q-1$-th root which is a one unit in $\mathbf{F}_q((1/T))$. A similar remark applies to $\overline{\tilde{\pi}}^{1/(q-1)}$.

Since $-1 = \sum (q-1)q^i$, from the definitions of section 1.2, we have

$$\overline{\Gamma}(0) = \overline{\Pi}(-1) = \lim \overline{(D_0 \cdots D_n)^{q-1}}$$

It is easy to see from the definition of D_i, that

$$D_0 \cdots D_n = D_{n+1}/[1]\cdots[n+1]$$

Hence

$$\overline{\Gamma}(0)^{q-1}/\overline{\tilde{\pi}^{q-1}} = \lim \overline{D_{n+1}^{q-1}}/\overline{[1]^{q^{n+1}}} = 1$$

since we have already seen that $\overline{D_i} \to 1$, and since $\overline{[1]}^{q^n} \to 1$, because any one unit raised to the q^n-th power tends to 1 as $n \to \infty$.

Hence we have proved $\overline{\Gamma}(0) = \overline{\tilde{\pi}}$. Observing that $a/(1-q) = \sum aq^i$ for $0 \le a \le q-1$, we get

Theorem 2.1 *For $0 \le a \le q-1$, we have*

$$\overline{\Gamma}(1 - \frac{a}{q-1}) = (\overline{\tilde{\pi}})^{a/(q-1)}$$

In particular, if $q \neq 2^n$, then

$$\overline{\Gamma}(1/2) = \sqrt{\overline{\tilde{\pi}}}$$

Corollary 2.2 $\overline{\Gamma}(i - a/(q-1)), \overline{\Gamma}(n)$ *are transcendental, where a, i, n are integers, $0 < a < q-1$ and $n \le 0$.*

This follows from the fact that

$$\overline{\Gamma}(z+1)/\overline{\Gamma}(z) = \overline{[1]\cdots[n]}$$

if q^n divides z, and q^{n+1} does not.(This excludes only $z = 0$). This is immediate from the definitions together with the observation made above that $D_0 \cdots D_n = D_{n+1}/([1]\cdots[n+1])$.

To investigate the nature of gamma values at fractions, in general, similarly, it is sufficient to look at $\overline{\Pi}(q^j/(1-q^t))$ for $0 \leq j \leq t$. They can be related to the periods $\tilde{\pi}_t$'s of Carlitz modules for $\mathbf{F}_{q^t}[T]$. For example,

$$\begin{aligned}\frac{\overline{\Pi}(1/(1-q^t))}{\overline{\Pi}(q^{t-1}/(1-q^t))^q} &= \lim \overline{\frac{D_{tn}D_{t(n-1)}\cdots D_0}{D_{tn-1}{}^q \cdots D_{t-1}{}^q}} \\ &= \lim \overline{[tn][t(n-1)]\cdots[t]} \\ &= (\overline{\tilde{\pi}_t})^{-1}\end{aligned}$$

Similarly, it can be shown, for example, that

$$\overline{\Pi}(1/(1-q^2))^{q^2-1} = \overline{\tilde{\pi}^q \tilde{\pi}_2{}^{-(q-1)}}$$

$$\overline{\Pi}(q/(1-q^2))^{q^2-1} = \overline{\tilde{\pi}\tilde{\pi}_2{}^{q-1}}$$

and it is easy to prove from this an analogue of Chowla-Selberg formula in the case of quadratic 'imaginary' extension $\mathbf{F}_{q^2}(T)$ of $\mathbf{F}_q(T)$. (Denominators are q^2-1 instead of the 'discriminants', since the extension is obtained by adjoining the (q^2-1)-st roots of unity). Also transcendence questions about these values get transferred to the transcendence questions for certain combination of these periods, which one might be able to handle by techniques of Yu [Yu 1].

2.2 Multiplication and reflection formulae

Recall that the classical gamma function satisfises (1)Reflection formula: $\Gamma(z)\Gamma(1-z) = z/\sin \pi z$ and (2)Multiplication formula:

$$\Gamma(z)\Gamma(z+\frac{1}{n})\cdots\Gamma(z+\frac{n-1}{n})/\Gamma(nz) = (2\pi)^{(n-1)/2}n^{1/2-nz}$$

We will prove analogues of these. The proof naturally falls into two parts. The first one, which is the subject of section 1,3,4; evaluates '$\Gamma(0)$' and the other part, which is the subject of this section, establishes relations between gamma values in the abstract setting below.

Consider a function f defined on $\mathbf{Z}_p$ by

$$f(\sum a_j q^j) = \prod A_j^{a_j}$$

for some A_j's. You can think of A_j's as independent variables with the evident manipulation rules. Put $g(z) = f(z-1)$. The various factorial('f') and gamma('g') introduced in this thesis are all of this form.

We want to get formal relations satisfied by f. In particular, we would like to know when $\prod f(x_i)^{n_i} = 1$ formally i.e. independently of A_i's. In other words, if x_i has q-adic expansion $\sum x_{ij} q^j$, then we want to know about the kernel of the map

$$\sum_{formal} n_i(x_i) \to (\sum_i n_i x_{ij})_j$$

If $\sum n_i(x_i)$ is in the kernel, then

$$\sum n_i x_i = \sum n_i \sum x_{ij} q^j = \sum_j (\sum_i n_i x_{ij}) q^j = 0$$

Lemma 2.3 *For $z \neq 0$, $g(z+1)/g(z)$ depends only on* $\mathrm{ord}_q(z)$.

This is obvious from the definition.

Claim: $g(z)g(1-z) = g(0)$: This follows (replace z by $z+1$) since $-1 = \sum (q-1)q^j$ and if $z = \sum z_j q^j$ then $-1-z = \sum (q-1-z_j)q^j$. Another way to prove this is to notice that $g(1) = f(0) = 1$, so it is true for $z = 0$. And since $\mathrm{ord}_q(z) = \mathrm{ord}_q(-z)$ by the lemma we have $g(z+1)/g(z) = g(1-z)/g(-z)$, hence by induction it is true for all integers z. Integers being dense in $\mathbf{Z}_p$, the claim follows easily.

Claim: $g(z)g(z+\frac{1}{n})\cdots g(z+\frac{n-1}{n})/g(nz) = g(0)^{(n-1)/2}$ for $(n,q) = 1, z \in \mathbf{Z}_p$. (Here, if n is even, so that q is odd, then we mean by $g(0)$ the element $\prod_{j=0}^{\infty} A_j{}^{(q-1)/2}$ whose square is $g(0)$). : Since $\mathrm{ord}_q(z) = \mathrm{ord}_q(nz)$, the lemma implies that

$$\frac{g(z)g(z+\frac{1}{n})\cdots g(z+\frac{n-1}{n})}{g(nz)} \cdot \frac{g(nz+1)}{g(z+\frac{1}{n})g(z+\frac{2}{n})\cdots g(z+1)}$$

$$= \frac{g(z)}{g(z+1)} \cdot \frac{g(nz+1)}{g(nz)} = 1$$

and again as before it is enough to prove the claim for a single z, say $z = 1/n$. So we want to prove $g(1/n)\cdots g((n-1)/n) = g(0)^{(n-1)/2}$, which follows from the reflection formula proved above, by pairing $g(a/n)$ with $g((n-a)/n)$.

It is amusing to note that it also follows immediately from the well-known results (eg. Hardy and Wright, chapter 9) on digit expansions, namely :If $(n, q) = 1$, $-1/n$ has a purely recurring expansion of r reccuring digits where r is minimal such that n divides $q^r - 1$ and essentially the reccuring digits for $-a/n$ are just obtained by permutations of those of $-1/n$, so that the sum of the i-th digits of all of them is constant, which then is easily seen to be $(q-1)(n-1)/2$, as $-1/n + \cdots + -(n-1)/n = -(n-1)/2 = ((n-1)/2)\sum(q-1)q^j$.

Hence we have proved

Theorem 2.4 *(1)* $g(z)g(1-z) = g(0)$
(2) $g(z)g(z+\frac{1}{n})\cdots g(z+\frac{n-1}{n})/g(nz) = g(0)^{(n-1)/2}$

We will prove a much more general result giving the formal relations in the next chapter. Here we end this section by recording some simple useful observations, which will be used without specific mention henceforth.

(1) $g(z) = g(0)/g(1-z) = g(0)/f(-z)$
(2) $f(\sum a_j q^j) = \prod f(q^j)^{a_j} \quad 0 \le a_j \le q-1$
(3) Since $1/(1-q^r) = 1 + q^r + \cdots$,

$$f(\sum_{j=0}^{r-1} a_j q^j/(1-q^r)) = \prod f(q^j/(1-q^r))^{a_j}$$

(4)

$$\prod_{j=0}^{r-1} f(\frac{q^{jh}}{1-q^{hr}}) = f(\frac{1}{1-q^h})$$

2.3 The general case

In this section we define the gamma function in general and relate its value at 0 to the period of the appropriate Drinfeld module.

Let K be a function field of one variable with its field of constants $k = \mathbf{F}_q$. Let ∞ be a place of K and δ be its residue class degree, A be the ring of integers outside ∞, K_∞ be the completion of K at ∞, k_∞ be the residue field $\mathbf{F}_{q^\delta}$ of ∞, and let Ω be the completion of an algebraic closure of K_∞.

For z in $K_\infty^\times$, define degree by $\deg z = -\delta \cdot \mathrm{ord}_\infty(z)$, so that for x in A, $\deg x$ is just the usual notion i.e. $\deg x = \dim_{\mathbf{F}_q}(A/xA)$ = degree of the divisor of zeroes of x. Notice that degree is always a multiple of δ . For an ideal $\mathcal{A}$ of A , put

$$\mathcal{A}_N = \{a \in \mathcal{A} : \deg a \leq N\}.$$

Then the Riemann part of the Riemann-Roch theorem shows that $\#\mathcal{A}_{i\delta} = q^{i\delta+c}$ for $i \in \mathbf{N}$, $i >> 0$, c some constant.

Choose a uniformizer t at ∞. This gives us splitting of $K_\infty^\times$ as $\mathbf{F}_{q^\delta}^\times \times U^1 \times t^{\mathbf{Z}}$ where U^1 is the group of one-units at ∞ . In other words, z in $K_\infty^\times$ can be written uniquely as $z = \mathrm{sgn}(z) \times \overline{z} \times t^n$,with $\mathrm{sgn}(z) \in k_\infty^\times, \overline{z} \in U^1, n \in \mathbf{Z}$.

This gives a homomorphism sgn : $K_\infty^\times \rightarrow k_\infty^\times$ such that $\mathrm{sgn}(U^1) = 1, \mathrm{sgn}(t) = 1, \mathrm{sgn}(a) = a$ for $a \in k_\infty^\times$. There are $q^\delta - 1$ such sgn functions, depending on choice of t. For $\sigma \in Gal(k_\infty/\mathbf{F}_q)$, $\sigma \circ \mathrm{sgn}$ is called a twisted sign function and a Drinfeld module ρ is called sign-normalised (see [Ha 1] p. 224), if the leading coefficient of ρ is a twisting of sign.

By [Ha 1] p. 224, every rank 1 Drinfeld A-module of generic characteristic over Ω (i.e. such that the map $x \mapsto D(\rho_x)$, where $D(\rho_x)$ is the constant term of ρ_x, is the inclusion $A \hookrightarrow \Omega$) is isomorphic to a sign-normalised ρ , for chosen sgn.

So choose a sgn function and let ρ be a corresponding sign-normalised rank one Drinfeld A-module of generic characteristic with corresponding rank one lattice Λ and exponential (or sine) $e_\rho = e_\Lambda$. Since A is a Dedekind domain, Λ is isomorphic to an ideal, say $\mathcal{A}$, of A. (Choose such an $\mathcal{A}$, and let $\tilde{\pi} \in \Omega$ be a corresponding 'period' defined up to an element in $\mathbf{F}_q^\times$ by the equation $\Lambda = \tilde{\pi}\mathcal{A}$. Think, if you will, of this period $\tilde{\pi}$ of ρ as an anologue of period $2\pi i$ (up to ± 1) in the situation $\mathbf{Z} \hookrightarrow \mathrm{End} G_m$.

In this setting, Hayes and Gekeler ([Ha 1] p.233, [Ge 1] p. 36) have generalised Carlitz's formula for the period. This formula will be described and reinterpreted as the identity $\Gamma(0) = \tilde{\pi}$ multiplied by a root of unity.

Let x be an element of A of degree > 0 and with $\mathrm{sgn}(x) = 1$. Then, since ρ is sign-normalised of generic characteristic, we have

$$\rho_x(u) = u^{q^d} + \cdots + xu$$

For each $a \in \mathcal{A}$ mod $x\mathcal{A}$, $e_\rho(a\tilde{\pi}/x) = \tilde{\pi}e_{\mathcal{A}}(a/x)$ is an xA-torsion point of Ω viewed as A-module via ρ. (Since

$$\rho_x^\Lambda(e_\rho(u)) = e_\rho(xu)$$

we have

$$\rho_x^\Lambda(\tilde{\pi} e_{\mathcal{A}}(a/x) = \rho_x^\Lambda(e_{\tilde{\pi}\mathcal{A}}(\tilde{\pi} a/x) = e_\Lambda(\tilde{\pi} a) = 0 \ \ )$$

In other words, this follows from the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "a"] & \Omega \arrow[r, "e_\Lambda"] \arrow[d, "a"] & \Omega \arrow[r] \arrow[d, "\rho_a^\Lambda"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & \Omega \arrow[r, "e_\Lambda"] & \Omega \arrow[r] & 0
\end{tikzcd}$$

So these are distinct roots of $\rho_x(u)$. Since $(-1)^{q^d-1} = 1$ in characteristic p, comparison of the coefficients of t gives

$$x = \prod_{a \in \mathcal{A}/\mathcal{A}x} {}'\tilde{\pi} e_{\mathcal{A}}(a/x) = \tilde{\pi}^{q^d-1} \prod {}' e_{\mathcal{A}}(a/x)$$

So

$$\tilde{\pi}^{1-q^d} = \frac{1}{x} \prod_{a \in \mathcal{A}/\mathcal{A}x} {}' e_{\mathcal{A}}(a/x)$$

This implies $\tilde{\pi}^{q^d-1} \in K_\infty$. We will see later that $\tilde{\pi}^{q^\delta-1} \in K_\infty$.
Hence

$$\tilde{\pi}^{1-q^d} = 1/x \prod_{a \in \mathcal{A}/\mathcal{A}x} (a/x) \prod_{b \in \mathcal{A}} (1 - a/xb) = x^{-q^d}(\prod a \prod (xb - a)/xb) \qquad (*)$$

which is the limit of the same expression with $b \in \mathcal{A}$ replaced by $b \in \mathcal{A}_{N\delta}$ as N tends to infinity. (This is true, since the corresponding statement is true for the exponential). Now $(xb - 0)/xb = 1$, so if N is large, then the numerator of ( ), with $xb - 0$ allowed, is just the product of nonzero elements of $\mathcal{A}_{N\delta+d}$, whereas the denominator is $x^{(\#\mathcal{A}_{N\delta}-1)*q^d}$ times the q^d-th power of the product of nonzero elements of $\mathcal{A}_{N\delta}$. (As there are q^d a's).

Take the one-unit part of both sides and notice that

$$\overline{x}^{-q^d} * \overline{x}^{-(\#\mathcal{A}_{N\delta}-1)*q^d} = \overline{x}^{-\#\mathcal{A}_{N\delta}} = \overline{x}^{-q^{N\delta+c}} \to 1$$

since a one-unit raised to the q^r-th power tends to 1 as $r \to \infty$.

Hence,

$$\overline{\tilde{\pi}^{1-q^d}} = \overline{(\prod_{a \in \mathcal{A}_{N\delta+d}} {}'a)} / \overline{(\prod_{b \in \mathcal{A}_{N\delta}} {}'b)^{q^d}}$$

Keeping in mind that we want to interpret this as

$$\overline{\Gamma}(0)^{1-q^d} = \overline{\Pi}(-1)^{1-q^d} = ((\overline{D_0 D_1}\cdots)^{q-1})^{1-q^d}$$

suggests the following definitions for D_i, $\overline{D_i}$, $\overline{\Gamma}$:

First notice that, the field of constants being $\mathbf{F}_q$, if $x \in \mathcal{A}_i$ then $ax \in \mathcal{A}_i, a \in \mathbf{F}_q^\times$, so the signs (elements of $\mathbf{F}_q^\times$) appear in $\mathbf{F}_q^\times$- equivalence classes. So choose representatives for $\mathbf{F}_{q^\delta}^\times/\mathbf{F}_q^\times$ and let D_i be the product of all elements a of $\mathcal{A}$ of degree $i\delta$ with $\operatorname{sgn}(a)$ being one of these representatives. So $D_i \in \mathcal{A} \subset$ A. Also let d_i be the number of these elements. $\overline{D}_i$, the one-unit part, is obviously independent of the choice of representatives.(Notice that for $\delta = 1$ and $1 \in \mathbf{F}_q^\times/\mathbf{F}_q^\times$ as the representative $D_i =$ product of all monic (i.e. with sign 1) elements in A of degree i).

With these definitions, our equation becomes

$$\overline{\tilde{\pi}^{1-q^d}} = \lim_{N\to\infty} (\overline{D_0}\cdots\overline{D_{N+d/\delta}})^{q-1}/((\overline{D_0}\cdots\overline{D_N})^{q-1})^{1-q^d}$$

So
$$\lim(\overline{D_{N+d/\delta}}^{q-1}/(\overline{D_N}^{q-1})^{q^d} = 1$$

aracteristic p, q^d power spreads out the power series expansion,
But in chara a one-unit, we get $\overline{D_i}^{q-1} \to 1$, so $\overline{D_i} \to 1$.
so $\overline{D_i}$ being a one-unit, we get $\overline{D_i}^{q-1} \to 1$, so $\overline{D_i} \to 1$.

Hence $\overline{\Gamma} : \mathbf{Z}_p \to K_\infty$ given by

$$\overline{\Gamma}(1 + \sum \alpha_i q^i) = \prod \overline{D_i}^{\alpha_i}$$

is well defined and

$$\overline{\tilde{\pi}^{1-q^d}} = \overline{\Gamma}(0)^{1-q^d}$$

We will prove below that

Lemma 2.5

$$\gcd\{q^d - 1 : d = \deg(x),\ x \in A,\ \operatorname{sgn}(x) = 1\} = q^\delta - 1$$

Hence

$$\overline{\tilde{\pi}^{1-q^\delta}} = \overline{\Gamma}(0)^{1-q^\delta}$$

If $\tilde{\pi}$ is that $1 - q^\delta$-th root of $\overline{\tilde{\pi}^{1-q^\delta}}$ which is one-unit then $\overline{\Gamma}(0) = \overline{\tilde{\pi}}$.

E.U. Gekeler showed me how to interpret the degree as deg Γ (or logarithm gamma). Analysing the degrees on the both sides of the equation, we get

$$\begin{aligned}\deg \tilde{\pi}^{q^d-1} &= \lim(q^d \sum_{b\in\mathcal{A}_{N\delta}} {}'(\deg b) - \sum_{a\in\mathcal{A}_{N\delta+d}} {}'(\deg a) + d\#\mathcal{A}_{N\delta}) \\ &= (q^d-1)\sum (q-1)i\delta d_i\end{aligned}$$

where $d_i = (q^\delta - 1)q^{i\delta+c}/(q-1) \to 0$ q-adically as i tends to infinity(by Riemann's thm.).

This implies that the map $\mathbf{N} \to \mathbf{Z}$ given by $i \to \deg \Pi(i)$ interpolates to a continuous function $\deg \Pi : \mathbf{Z}_p \to \mathbf{Z}_p$ given by $\sum \alpha_i q^i \to \sum i\delta\alpha_i d_i$. Since $-1 = \sum (q-1)q^i$, we get $\deg\Pi(-1) = \sum (q-1)i\delta d_i$. So $\deg \Pi(-1) = \deg \tilde{\pi}$ and hence if we change the range of the gamma function, so as to make sense out of p-adic integral powers of t, we would get $\Pi(-1) = \tilde{\pi}$ times a root of unity, if we define $\Pi(z) = \overline{\Pi}(z)t^{-\deg \Pi(z)/\delta}$. These would be then independent of the choice of the uniformizer, if it gives the same sign function.

The most natural way to do this is to complete $K_\infty^\times$ p-adically, i.e. define $\hat{K}_\infty^\times = \varprojlim K_\infty^\times / K_\infty^{\times p^n}$. Since finite fields are perfect, signs die in it. We obtain a gamma funtion which is independent of the choice of sign.

If $\delta > 1$, as Prof. Tate pointed out to me, one can do a little better. Since d_i, for large i, is not only divisible by large power of q, but also by $(q^\delta - 1)/(q-1)$, we can put $\tilde{K_\infty}^\times = \varprojlim K_\infty^\times / K_\infty^{\times (q^\delta - 1)p^n/(q-1)}$ and take it as the range, in the evident fashion. The signs survive now in $\mathbf{F}_{q^\delta}^\times / \mathbf{F}_q^\times$.

Before analysing the root of unity (i.e. the sign from (*)) and the question of the variation of the situation with respect to the choices we make, we first prove the lemma above.

Proof of the lemma : It is enough to prove that gcd of d's is δ. First of all, by Riemann's theorem we know that gcd of degrees of elements of A is δ. Next, since K is dense in K_∞, there is an element of degree 0 in K of any given sign. Multiplying by elements in $\mathcal{A}$ of high degree, clearing the denominators we see that $\mathcal{A}$ has, for some large i, elements of degree i of all signs. Now, choose elements x_k's in A of degrees d_k such that gcd of d_k's is δ. Then multiplying by powers of x_k, we get all signs in degree $i + nd_k$ for all positive n, so that the gcd in question divides the gcd of $i + nd_k$ which is δ and the lemma is proved.

Now we analyse the signs in (*). The sign of the right hand side is the limit as $N \to \infty$ of the sign of the product of nonzero elements of $\mathcal{A}_{N\delta+d}$ divided by the product of nonzero elements of $\mathcal{A}_{N\delta}$. (As q^d-th power is identity on $\mathbf{F}_{q^\delta}$). Varying d, a simple gcd arguement then shows that $\tilde{\pi}^{1-q^\delta} = \epsilon\Gamma(0)^{1-q^\delta}$ where the sign ϵ is the stationary limiting sign (we have shown that it exists, for a more direct proof see [Gel] pa.30) of the product of elements of $\mathcal{A}$ of degree $N\delta$ as $N \to \infty$. If $\delta = 1$, as $\prod_{a \in \mathbf{F}_q^\times} a = -1$, and as $(-1)^{q^d} = -1$ it follows by straightforward counting using Riemann's theorem that $\epsilon = -1$. Hence,

$$\Gamma(0)^{q-1} = -\tilde{\pi}^{q-1} \in (K_\infty^\times)_{sgn=1} \subset \hat{K}_\infty^\times$$

as $(q-1)\deg\Gamma(0) \in \mathbf{Z}$ rather than just in $\mathbf{Z}_p$.

We summarise the discussion in

Theorem 2.6

$$\Gamma(0) = \mu\tilde{\pi}$$

(where μ is at most $(q^\delta - 1)^2$-th root of unity. If $\delta = 1$, it is $(q-1)$-th root of -1), in the sense that '$q^\delta - 1$-th powers of the both sides' are the same.

Jing Yu ([Yu 1] thm. 5.1) has proved transcendence of $\tilde{\pi}$, which then implies transcendence of some special values of the gamma function as in section one.

It should be noted that in the case $A = \mathbf{F}_q[T]$ we studied in section one, Γ and ρ did not correspond rightly from the point of view of this section. If uniformizer T corresponds to sign function sgn and for $a \in \mathbf{F}_q^\times$, T/a corresponds to sgn' say, then $\rho_T = T + F$, $\rho_T{}' = T + aF$ are sgn-normalised for sgn and sgn' respectively. So the Carlitz module is sgn-normalised for uniformizer $-T$ rather than T. Now ρ and ρ' are isomorphic. Suppose the correponding lattices are Λ and $\mu\Lambda$. Then $\mu^{-1}(T + aF)\mu = (T + F)$ i.e. $\mu^{q-1}a = 1$. In other words, in this case, change of sign function changes $\tilde{\pi}$ up to a $(q-1)$-th root of an element in $\mathbf{F}_q^\times$.

To summarise, given a rank one Drinfeld-module, we get an ideal class in A. If we choose an ideal $\mathcal{A}$ representing the class, we get a gamma function and if in addition we choose a sgn function we get $\tilde{\pi}$ defined upto $\mathbf{F}_{q^\delta}^\times$ and $\Gamma(0) = \tilde{\pi}$ times a root of unity(at most $(q^\delta - 1)^2$-th, at most $2(q-1)$-th if $\delta = 1$). Observe that, if $q = 2$ and $\delta = 1$, the question of signs disappears. A change of sgn function may change $\tilde{\pi}^{q^\delta - 1}$ by element in $\mathbf{F}_{q^\delta}^\times$, but leaves the formula intact. The Γ function is independent of choices of coset representatives of $\mathbf{F}_{q^\delta}^\times / \mathbf{F}_q^\times$, and it depends on the uniformizer only through its sign. If t and t' are uniformizers with $t = at'$, $a \in \mathbf{F}_{q^\delta}^\times$, then $a^{\deg \Gamma / \delta} \Gamma = \Gamma'$, so there are $(q^\delta - 1)/(q-1)$ gamma functions (hence unique if $\delta = 1$) and the gamma function is independent of the sgn choice on a smaller disc of $\mathbf{Z}_p$.

2.4 Interpolations at the finite places

In this section we show that the gamma function interpolates at all finite places v of A which are relatively prime to $\mathcal{A}$ and that $\Gamma_v(0) = (-1)^{\deg v - 1}$.

(Finite place :$\mathcal{P} = v$) : Let v be a finite place of A relatively prime to $\mathcal{A}$, with residue class degree h. We form $\tilde{D}_i = D_{i,v}$ as usual by removing the factors divisible by v : $\tilde{D}_i$ = product of elements a of degree $i\delta$ with sgn(a) one of the chosen representatives and $v(a) = 0$. Let us denote by S_i the set of these elements.

We will prove $(-1)^\delta \tilde{D}_i \to 1$, so put

$$\Pi_v(\sum \alpha_i q^i) = \prod((-1)^\delta \tilde{D}_i)^{\alpha_i}$$

so that $\Pi_v : \mathbf{Z}_p \to K_v$. Even though D_i depends on a choice of representatives for $\mathbf{F}_{q^\delta}^\times / \mathbf{F}_q^\times$, $\tilde{D}_i$ for large i does not, because the number of elements in S_i of given sign is a multiple of $q^h - 1$ and $q^h - 1$-th power kills the choice. Similarly it is independent of the choice of sgn for large i. So Γ_v is again unique on a smaller disc of $\mathbf{Z}_p$. In any case, a value $\Gamma_v(z)$ is determined up to multiplication by an element in $\mathbf{F}_q^\times$. (So again, there is a unique function for $q = 2$). And we may want to normalise.

$(-1)^\delta \tilde{D}_i \to 1$: Let $i >> 0, i > h, w = [i/h]$. Claim :$(-1)^\delta \tilde{D}_i \equiv 1 \bmod v^w$: Product of elements of $(\mathcal{A}/\mathcal{A}v^w)^\times$ is -1 mod v^w, by generalisation of Wilson's theorem and elements of S_i are equidistributed among the cosets, since elements of S_i with fixed sign are. (This is because, for large i, all members of one coset can be transfered to any other by subtracting coset-representatives of small degree, leaving the sign unchanged). So enough tpt number of elements of S_i in any coset is $\equiv \delta$ mod 2. Now $\#\{a \in \mathcal{A} : \deg a \leq k\delta\} = q^{k\delta + c}$, by Riemann's theorem,so that $\#\{a \in \mathcal{A} : \deg a = i\delta$ with sgn(a) one of the chosen$\} = (q^\delta - 1)/(q-1) q^{i\delta + c}$. So $\#S_i = ((q^\delta -$

$1)/(q-1))q^{i\delta-h+c}(q^h-1)$, but the number of cosets is $(q^h-1)q^h$, so that the required number is

$$\{(q^\delta-1)/(q-1)\}q^r \equiv (q^\delta-1)/(q-1) \equiv \delta \bmod 2$$

as claimed. (If $p=2$, nothing to prove)

Now we will prove

$$\Pi_v(-1) = (-1)^{h-1}$$

This boils down to proving

$$(\prod_{m \in S_0 \cup \cdots \cup S_i} m)^{q-1} \equiv (-1)^{h-1}$$

mod v^l with $l \to \infty$ as $i \to \infty$. We will prove this with $i = lh-1$. (We already know that the limit exists). : Since v is of degree h and since $\{am\}$ as a runs through $\mathbf{F}_q^\times$ spans the reduced residue class system mod v^l, we have

$$-1 \equiv (\prod a)^{\# S_0 \cup \cdots \cup S_i}(\ldots)^{q-1}$$

But $\prod a = -1$, so tpt $\# S_0 \cup \cdots S_{lh-1} \equiv h \bmod 2$. But

$$\#\{m\} = (q^{lh-1+c} - q^{lh-1-h+c})/(q-1) \equiv (q^h-1)/(q-1) \equiv h \bmod 2$$

(If $p=2$, nothing to prove).

Hence we have proved

Theorem 2.7 $\Gamma_v(0) = (-1)^{\deg v-1}$ *for all* v *prime to* $\mathcal{A}$. *In particular, for* $0 \le a \le q-1$ $\Gamma_v(1-\frac{a}{q-1})$ *are roots of unity and* $\Gamma_v(\frac{b}{q-1})$ *is algebraic for* $b \in \mathbf{Z}$.

Observe that the algebraicity statement here is a very special case of analogue of Gross-Koblitz algebraicity result [GK] pa.571, cor.1.8.

Chapter 3

Gauss sums for function fields

In this chapter, we propose a definition of Gauss and Jacobi sums and prove analogues of many classical and recent results about them, including those of Stickelberger, Hasse-Davenport, Weil, Gross-Koblitz and Deligne. For comparison, one should consult [GK] and chapter one of Lang's *Cyclotomic fields*.

The first section computes some values of the v-adic gamma function. These values enable us to compute one side of the Gross-Koblitz formula. The second section defines Gauss and Jacobi sums and develops their elementary properties. In the third section, we prove analogues of the Stickelberger and Gross-Koblitz theorems and get Stickelberger elements for constant field extensions. In contrast to the Gross-Koblitz situation, here the proof is direct and does not use the analogue of Katz' computation involving p-adic cohomology of Fermat curves. The last section contains the proof of analogues of Weil's theorem on Jacobi sums as Hecke characters, Deligne's theorem and remarks on the general situation.

3.1 The gamma side

Let P be a monic irreducible polynomial in T representing a prime of degree h of $A = \mathbf{F}_q[T]$, and let $0 \leq r \leq h-1$. Let

$$M_r = \Pi_P(\frac{q^r}{1-q^h}) = (-\tilde{D}_r)(-\tilde{D}_{r+h})(-\tilde{D}_{r+2h})\cdots$$

Since $\tilde{D}_a$ is formed by removing n divisible by P from the product $D_a = \prod n$ (n runs through the monic elements of degree a of A), we have $\tilde{D}_a = D_a/(D_{a-h}P^l)$ where l is such that $\tilde{D}_a$ is unit at P. Hence

$$\begin{aligned}\tilde{D}_r \cdots \tilde{D}_{r+mh} &= \frac{D_r \cdots D_{r+mh}}{D_r \cdots D_{r+(m-1)h}P^w} \\ &= \frac{D_{r+mh}}{P^w} \\ &= T_{r+mh}\end{aligned}$$

say. (Here w is $\mathrm{ord}_P D_{r+mh}$). Then $M_r = \lim(-1)^{m+1}T_{r+mh}$.

Since $D_i = [i]D_{i-1}{}^q$,

$$T_{r+mh} = [r+mh][r-1+mh]^q \cdots \frac{[mh]^{q^r}}{P^{q^r}} \cdots [r+1+(m-1)h]^{q^{h-1}} T_{r+(m-1)h}{}^{q^h}$$

because $[l]$ is product of all monic primes of degree dividing l, so that P divides $[l]$ (and then divides it only to the first power) iff h divides l. Hence

$$M_r = [\ ][\ ]^q \cdots \frac{[\ ]^{q^r}}{P^{q^r}} \cdots [\ ]^{q^{h-1}} M_r{}^{q^h}$$

Now $[l+mh] = T^{q^{mh+l}} - T$. Wlg. assume $P \neq T$ (otherwise replace T by $T+1$). Let $T = au$ be the decomposition in the completion K_P of T, a unit at P, as product of its 'Teichmuller representative' a and its one unit part u. As $a^{q^{mh}} = a$ and $u^{q^t} \to 1$ as $t \to \infty$, we have, as $m \to \infty$,

$$[l+mh] = T((au)^{q^{mh+l}-1} - 1) \to (a^{q^l} - T)$$

Hence

$$M_r{}^{1-q^h} = -\frac{(a^{q^r} - T)(a^{q^{r-1}} - T)^q \cdots (a^{q^{r+1}} - T)^{q^{h-1}}}{P^{q^r}}$$

Let $\mathbf{F}_{q^h} = \mathbf{F}_q(a)$. In $\mathbf{F}_{q^h}[T]$, P splits into monic linear factors (monic representatives of primes above P)

$$\mathcal{P}_1 = (T-a), \mathcal{P}_2 = (T - a^{q^{h-1}}), \cdots \mathcal{P}_h = (T - a^q)$$

(The inclusion $\mathbf{F}_{q^h}(T) \subset K_P$ induces valuation v on K_P s.t. $v(\mathcal{P}_i)$ is 1, if $i = 1$ and 0 otherwise) So, if σ is the Frobenius (q-th power) element of $\mathrm{Gal}(\mathbf{F}_{q^h}(T)/\mathbf{F}_q(T))$, $\mathcal{P}_r = \sigma^{-r+1}\mathcal{P}_1$. ($r$ modulo h) Hence,

$$M_r{}^{q^h-1} = \frac{(-P)^{q^r}}{(-\mathcal{P}_{1-r})(-\mathcal{P}_{2-r})^q \cdots (-\mathcal{P}_{-r})^{q^{h-1}}} = (-\mathcal{P}_1)^{t_r}$$

where $t_r = \sum_{i=0}^{h-1}(q^r - q^i)\sigma^{r-i}$. So M_r is determined up to a $q^h - 1$-th root of unity, which can be effectively determined by looking at signs of $-\tilde{D}_i$ for small i. Notice that M_r generates a Kummer extension of the cyclotomic extension $\mathbf{F}_{q^h}(T) = \mathbf{F}_q(T)(\mu_{q^h-1})$ of $\mathbf{F}_q(T)$. Also, multiplying out M_r's, we see that $\Pi_P(1/(1-q))^{q^h-1} = 1$, which is a special case of more precise result we have proved for general $\mathcal{A}$ in the last chapter. We have proved,

Theorem 3.1 *Let P be a prime of degree h of $\mathbf{F}_q[T]$ and let $\mathcal{P}_1 = (T-a)$, then*

$$\Pi_P(\frac{q^r}{1-q^h}) = (-\mathcal{P}_1)^{t_r/(q^h-1)}$$

for $0 \leq r \leq h-1$, with $t_r = \sum_{i=0}^{h-1}(q^r - q^i)\sigma^{r-i}$, and the $q^h - 1$-th root is the one congruent to $-D_r$ mod P.

Corollary 3.2 *If N divides $\mathbf{N}P - 1$, then $\Gamma_P(i/N)$ is algebraic for any integer i.*

In fact, we have an explicit expression for it. The corollary follows from the theorem, since we know $\Pi_P(z+1)/\Pi_P(z), \Pi_P(0), \Pi_P(-1)$, so it remains to consider fractions between 0 and 1 having denominators $q^h - 1$, and those are all obtained by multiplications (depending on the 'digits' of the numerator) of the basic values evaluated in the theorem.

Observe that we have proved an analogue of the algebraicity result of [GK] cor. 1.8 without using p-adic cohomology or the Gross-Koblitz formula. In fact, we can evaluate the rhs of [GK] thm. 1.7 (up to a $q^h - 1$-th root of unity, which depends on the choice of λ, which we will fix), in light of the remarks made at the end of the section 2.2.

3.2 Gauss and Jacobi sums

Classically, a Gauss sum is a certain sum of the form

$$-\sum_{x \in \mathbf{F}_{p^m}^\times} \chi(x)\psi(\mathrm{Tr}x)$$

where χ is a nontrivial multiplicative character $\chi : \mathbf{F}_{p^m}^\times \to \mu_{p^m-1}$ and ψ is a nontrivial additive character $\psi : \mathbf{F}_p \to \mu_p$ and Tr is the trace from $\mathbf{F}_{p^m}$ to $\mathbf{F}_p$.

Our key idea in making Gauss sums for function fields is to view ψ rather as an isomorphism of $\mathbf{Z}$-modules $\mathbf{Z}/(p) \rightarrow (\mathbf{G}_m)_p = \mu_p$ and in view of the analogy between the rank one situations $\mathbf{Z} \rightarrow \mathrm{End}\mathbf{G}_m$ and $\rho : A \rightarrow \mathrm{End}\mathbf{G}_a$, to replace it by an isomorphism of A-modules $A/P \rightarrow (\mathbf{G}_a)_P = \Lambda_P$ say. (We do not use the notation μ_P to avoid confusion.) Note that as a finite field, $A/P \cong \mathbf{F}_{\mathbf{N}P}$ and we use the trace from $\mathbf{F}_{\mathbf{N}P^m}$ to $\mathbf{F}_{\mathbf{N}P}$.

Now we proceed to define and develop the Gauss sums (in analogous fashion to [GK] pa. 570,571, but as we have been already commited to some notation, the notation will not be completely consistent with the analogy).

Let K, A, ∞, q be as in section 2.3. Let N be a positive integer prime to p and let $F = K(\mu_N)$ be the cyclotomic field of N-th roots of unity (a constant field extension, in our case). Let B be the integral closure of A in F. Let $\wp$ be a prime of B above a prime P of A of degree h, with relative residue class degree f, so that $\mathbf{N}\wp = \mathbf{N}P^f = q^{hf}$. Let r be the smallest positive integer such that N divides $q^r - 1$, so that the constant field in F is $\mathbf{F}_{q^r}$. Note that r divides hf. Let ρ be a sgn-normalised rank one Drinfeld A-module over the Hilbert class field H_A of A.(It has integral coefficients) (See [Ha 2] pa.205, following the theory developed in this paper, it seems that we can even use an order instead of the full ring of integers A). (For example, if $A = \mathbf{F}_q[T]$ Carlitz' module can be taken as ρ). Λ_P will denote P-torsion of ρ. Fix an isomorphism $\psi{:}k_P \rightarrow \Lambda_P$. Choose a prime $\underline{P}$ above $\wp$ in $K(\mu_{q^{hf}-1})$ and let t be the Teichmuller map from $k_{\underline{P}}{}^{\times}$ to the $q^{hf}-1$-th roots of unity in $K(\mu_{q^{hf}-1})$, which is the inverse to reduction mod $\underline{P}$. (We will denote analogous Teichmuller maps (eg. for P) by the same letter t.)

For $a \in (\frac{1}{N}\mathbf{Z})/\mathbf{Z} - \{0\}$,we define the Gauss sum $g(a, \wp)$ as follows: If $\sum_{j=0}^{hf-1} b_j q^j$ is the digit expansion of $b = a(q^{hf}-1)$ (denote the representative of a between 0 and 1 as $\langle a \rangle$ (or as a by abuse of notation) and think of b modulo $q^{hf} - 1$), then

$$g(a, \wp) = \prod_{j=0}^{hf-1} g(q^j)^{b_j} = g(b)$$

say, where

$$g(q^j) = -\sum t(x^{-q^j})\psi(\mathrm{Tr}x)$$

(for all $j > 0$), where the sum is over $x \in k_{\underline{P}}{}^{\times}$ and the Tr is the trace (from $k_{\underline{P}}$ to k_P). Notice that $g(a, \wp) = g(a, \underline{P})$ and $g(b) = g(q^j/(q^{hf} - 1), \underline{P})$.

For simplicity, assume that $\delta = 1$ and the class number is one.

A priori, $g(a,\wp)$ belongs to $K(\mu_{q^{hf}-1})(\Lambda_P)$. Now the powers of Frobenius (q-th power) σ for the extension $K(\mu_{q^{hf}-1})$ over K and elements $\mu \in (A/P)^\times = \mathrm{Gal}(K(\Lambda_P)/K)$ can be thought of, in the obvious fashion, as elements of the Galois group of $K(\mu_{q^{hf}-1})(\Lambda_P)$ over K. (As P is totally ramified in $K(\Lambda_P)$, $K(\mu_{q^{hf}-1})(\Lambda_P)$ is just the compositum of linearly disjoint extensions $K(\mu_{q^{hf}-1})$ and $K(\Lambda_P)$ of K). The basic sums are essentially 'Lagrange resolvents'.

If $\underline{a} = \sum m(a)\delta_a$ is an element of the free abelian group with basis $(\frac{1}{N}\mathbf{Z})/\mathbf{Z} - \{0\}$, we may define the generalised Gauss sum $g(\underline{a},\wp) = \prod g(a,\wp)^{m(a)}$. Also, for $u \in (\mathbf{Z}/(N\mathbf{Z}))^\times$, let $\underline{a}^{(u)} = \sum m(a)\delta_{ua}$. Put $n(\underline{a}) = \sum m(a)\langle a\rangle$.

Theorem 3.3 *(1) $g(\underline{a},\wp)$ is independent of the choice of $\underline{P}$.*
(2) $g(\underline{a},\wp)$ belongs to $L = F(\Lambda_P)$.
(3) $g(\underline{a},\wp)^\sigma = g(\underline{a}^{(q)},\wp)$
(4) $g(\underline{a},\wp)^\mu = t(\mu)^{n(\underline{a})(q^{hf}-1)}g(\underline{a},\wp)$
(5) When $n(\underline{a})(q^{hf}-1)/(q^h-1)$ is an integer, then $g(\underline{a},\wp)$ belongs to F. (In particular, if $n(\underline{a})$ is an integer, the 'Jacobi sum' $g(\underline{a},\wp)$ (eg. $g(a,\wp)^M$, where M is the denominator of a and $g(a,\wp)^{q^j-\sigma^j}$) belongs to F.
(6) $g(\underline{a},\wp)$ depends on the choice of ψ only up to multiplication by q^h-1-th root of unity, and is independent of ψ if $n(\underline{a})(q^{hf}-1)/(q^h-1)$ is an integer.
(7) $g(\underline{a}^{(q^h)},\wp) = g(\underline{a},\wp)$.

Remark: Our 'Gauss' and 'Jacobi' sums (By Jacobi sum we mean $g(\underline{a},\wp)$ when $n(\underline{a})$ is an integer, since this is precisely the case when classically they are character sums (or at least multiplicatively obtained from those) built out of the multiplicative characters as in Weil- Collected works, [1952d] pa. 488) are products of 'character' sums, by the definition, but $g(a,\wp)$ is not necessarily given by straight analogue of the classical case (i.e. (1.2) of [GK]), similarly since our basic multiplicative characters take values in the finite fields, whereas the values of the Jacobi sums do not necessarily lie there, the situation is different in that case too.

Proof: From the definition of $g(q^j)$ as a character sum and since the trace is linear, it is clear that $g(q^j)^\sigma = g(q^jq)$ and $g(q^j)^\mu = t(\mu)^{q^j}g(q^j)$ and the same holds for 'b' by the multiplicative nature of the definitions, and (3), (4) follow by multiplicativity from $g(a,\wp) = g(a(q^{hf}-1))$.(Remark: If $N = q^r - 1$,but not in general, $g((\sum b_jq^j)/N,\wp) = \prod g(q^j/N,\wp)^{b_j}$). As

$(q^{hf}-1)/(q^h-1) = 1+q^r+\cdots+q^{hf-r}$, (3) implies that $g(a,\wp)$ is invariant wrt. σ^r and (2) follows. Similarly, as $\underline{P}$'s above $\wp$ are conjugates by power of σ^r, and $g(b,(\underline{P})^{\sigma^r}) = g(b,\underline{P})^{\sigma^r}) = g(bq^r,\underline{P})$, (1) follows. Since $t(\mu)$ is a q^h-1-th root of unity, the hypothesis of (5) implies invariance wrt. μ action and so (5) holds. Since, any change in ψ can be accomplished by the action of μ, (6) follows from (4). As $\mathrm{Tr}x^{q^h} = \mathrm{Tr}x$ the substitution $y = x^{q^h}$ proves (7). (Observe that (2) can be impoved sometimes (eg. by (7))).

If we are willing to have the values of the Gauss sums in the completion $K_\wp$, then we can avoid using $\underline{P}$ and define

$$g^*(q^j/(\mathrm{N}\wp-1),\wp) = -\sum t(x^{-q^j})\psi(\mathrm{Tr}x)$$

where the sum is over $x \in {k_\wp}^\times$, t is the Teichmuller map to the completion and the trace is from $k_\wp$ to k_P. Define $g^*(a,\wp)$, now for a's with denominators $\mathrm{N}\wp-1$, in analogous fashion.

Theorem 3.4

$$-\sum_{x\in {k_\wp}^\times} t(x)^{-q^j}\psi(\mathit{Tr}x) = -\sum_{x\in {k_P}^\times} t(x)^{-q^j}\psi(x)$$

In fact, if h is a function on $\mathbf{F}_{q^f}$ with values in a ring containing $\mathbf{F}_{q^f}$ and with $h(0)=0$ then

$$\sum_{x\in\mathbf{F}_{q^f}^\times} x^{-q^j}h(\mathrm{Tr}x) = \sum_{y\in\mathbf{F}_q^\times}\sum_{\mathrm{Tr}x=y} x^{-q^j}h(y) = \sum_{y\in\mathbf{F}_q^\times} y^{-q^j}h(y)$$

because

$$\sum_{\mathrm{Tr}x=y} x^{-q^j} = y^{-q^j}$$

To see this, put $a_{j,f} = \sum x^{-q^j}$, for $y=1$. Then

$$\sum_{\mathrm{Tr}x=y} x^{-q^j} = y^{-q^j}\sum_{\mathrm{Tr}x=1} x^{-q^j} = y^{-q^j}a_{j,f}$$

$$-1 = \sum x^{-q^j}(\mathrm{Tr}x) = \sum_y{}'\sum_{\mathrm{Tr}x=y} x^{-q^j}y = a_{j,f}\sum{}'y^{1-q^j} = -a_{j,f}$$

implies that $a_{j,f} = 1$. This proves the theorem.

As a corollary, we see that

$$g^*(a,\wp) = g^*(\frac{\mathrm{N}\wp - 1}{\mathrm{N}P - 1}a, P)$$

So if $\tilde{\wp}$ is a prime above $\wp$ in a constant field extension of F, then

$$g^*(a,\tilde{\wp}) = g^*(\frac{\mathrm{N}\tilde{\wp} - 1}{\mathrm{N}\wp - 1}a, \wp)$$

Prof. Tate suggested me the following nice way to look at the definition of the basic sums: Let $\tilde{k}$ be a finite field of 'characteristic P' i.e. a finite extension of $k_P = A/P$. Let Φ be a $\mathbf{F}_q$-homomorphism Φ: $\tilde{k} \to L$ where L is a field containing $K(\Lambda_P)$. Then a basic sum is just

$$g(\Phi) = \sum_{x \in \tilde{k}^\times} \Phi(x^{-1})\psi(\mathrm{Tr}x)$$

where the trace is from $\tilde{k}$ to k_P.

Theorem 3.5 *(Analogue of the Hasse-Davenport theorem): If $\tilde{\wp}$ is a prime (in a constant field extension of F) of relative residue class degree s over $\wp$, then $g(\underline{a}, \tilde{P}) = g(\underline{a}, \wp)^s$*

This follows from the theorem 3.4 and (7) of the theorem 3.3, if we observe that $q^j/(q^h - 1) = q^j(1 + q^h + \cdots + q^{h(s-1)})/(q^{hs} - 1)$.

Let $\overline{P}$ be the unique prime over $\wp$ in $L = F(\Lambda_P)$ and let $L_{\overline{P}}$ be the completion of L at $\overline{P}$. Now $\psi(1) \in \Lambda_P - \{0\}$ (a 'primitive P-th root'). By abuse of notation, we also denote $\psi(1) = \Lambda_P$. Notice that $g(\underline{a}, \wp) \in K_P(\Lambda_P)$ by (7) of the theorem.

From now on, P, $\underline{P}$, $\wp$ etc. (the primes in the constant field extensions of K) will mean the monic representatives of the corresponding ideals.

Lemma 3.6 *There exists a unique $\lambda \in K_P(\Lambda_P) \subset L_{\overline{P}}$ such that $\lambda^{\mathrm{N}P-1} = -P$ and $\lambda \equiv \Lambda_P mod \Lambda_P^2$*

(I would like to thank Prof.Tate for telling me how to prove this lemma). Uniqueness is obvious. (By [Ca 2] thm. 11 or by [Ha 2] prop. 9.1) $\rho_P(u)/u$ is an Eisenstein polynomial $u^{\mathrm{N}P-1} + \cdots + P$ and so its root Λ_P generates a totally tamely ramified abelian extension of K_P of degree $\mathrm{N}P-1$, but any $\mathrm{N}P-1$-th root λ also generates such extension (Kummer extension) and since by local class field theory, such extensions being obtained by adjoining $\mathrm{N}P-1$-th root of prime elements, we have $K_P(\Lambda_P) = K_P((-\tilde{P})^{1/(\mathrm{N}P-1)})$ for some prime element $\tilde{P}$. Now from the equations we see that P and $\tilde{P}$ are norms from this extension and hence their ratio is a one-unit and hence $\mathrm{N}P-1$-th power. Hence the extension is same as $K_P(\lambda)$. This proves the lemma.

(λ can be thought of as a 'local Λ_P' for the Lubin-Tate action $T \to PT + T^{\mathrm{N}P}$)

Since ρ is of generic characteristic and with integral coefficients, we have

$$\psi(Tr(x)) = \rho_{Tr(x)}(\psi(1)) = \rho_{Tr(x)}(\Lambda_P) \equiv Tr(x)\Lambda_P \equiv Tr(x)\lambda \bmod \Lambda_P^2$$

$$g(1) \equiv -\lambda \sum t(x^{-1})Tr(x) \equiv \lambda \bmod \overline{P^2}$$

Hence we have proved

Proposition 3.7 $g(1) \equiv \lambda \mathit{mod} \overline{P^2}$ *In particular,* $g(1) \neq 0$, *so* $g(\underline{a}, \wp) \neq 0$.

This proposition is the first step of the proof of the Stickelberger theorem in the next section.

From now on, for simplicity, we assume that $A = \mathbf{F}_q[T]$, ($\underline{P}$ is just $T-a$ then, in the notation of the first section) and by Carlitz' module, we will mean $T \to T + F$. (Note the sign change)

Now we prove an analogue of the classical theorem :$g(\chi)g(\chi^{-1}) = \chi(-1)q$.

Theorem 3.8

$$\prod_{i=0}^{h-1} g(q^j)^{q-1} = (-1)^h P$$

In particular,$g(b)g(q^{hf}-1-b) = (-1)^{hf}P^f$ *and* $g(\underline{a}, \wp)$ *lies above* P.

Proof: (3), (4) of theorem 3.3 show that $(\prod_{i=0}^{h-1} g(q^j)^{q-1})$ is invariant by σ, μ action and hence belongs to A, as it is integral. Then the proposition implies that P divides it.

Lemma 3.9 $\deg g(q^j) = 1/(q-1)$.

: Enough to prove that $\deg \Lambda_P \leq 1/(q-1)$. Now the x-torsion points Λ_x are given by $\tilde{\pi} e_A(a/x)$, with $a \in A$, with $\deg a < \deg x$, hence $\deg \Lambda_x = \deg \tilde{\pi} + \deg(a/x) \leq \deg \tilde{\pi} - 1$. But by Carlitz' formula $\deg \tilde{\pi} = q/(q-1)$. Comparison of degrees then proves the lemma.

Hence the degrees of the basic sums (for any infinite valuation) are $1/(q-1)$ (analogue of all absolute values of classical Gauss sums being $q^{1/2}$) and that the theorem is true up to a root of unity. The following argument, suggested by Prof.Tate, fixes the root. Let $K(\Lambda_P)^+$ be the maximal 'totally real' subfield (i.e. the infinite place splits completely) of $K(\Lambda_P)$, then it is easy to see from the equations that the norms from $K(\Lambda_P)$ to $K(\Lambda_P)^+$, which are in $\mathbf{F}_q[T]$, are monic and the Galois group of $K(\Lambda_P)$ over $K(\Lambda_P)^+$ is $\mathbf{F}_q^\times$. But the norm of $g(1)$ to $K(\Lambda_P)^+$ differs, by (4) of theorem 3.3, from $\prod g(q^j)^{q-1}$ by

$$\prod_{\mu \in \mathbf{F}_q^\times} t(\mu)^{(q^h-1)/(q-1)} = (-1)^{(q^h-1)/(q-1)} = (-1)^h$$

This proves the theorem. We can think of this result as a reflection formula for the Gauss sums, which at the same time evaluates '$f(-1)$', in the notation of the section 2.2. Modified version of considerations in that section also proves a multiplication formula for the Gauss sums, just by digit considerations. This can also be recovered from the Gross-Koblitz formula in the next section.

3.3 Analogue of the Gross-Koblitz theorem

Theorem 3.10 *(For $\mathbf{F}_q[T]$ and $\rho_T = T + F$)*

$$g(a, \wp) = (-1)^{a(q^{hf}-1)}(-1)^{f(h-1)}\lambda^{(q^h-1)\sum_{i=0}^{f-1}\langle q^{hi}a\rangle} \prod_{i=0}^{f-1} \Gamma_P(\langle q^{hi}a\rangle)$$

in $L_{\overline{P}}$

We will in fact prove the stronger statement with $g(a, \wp)$ replaced by $g^*(a, \wp)$. (The class of a's is wider now)

This theorem is an analogue of the Gross-Koblitz theorem and will follow from the 'Stickelberger theorem' we will prove below and the computation of the gamma values in the first section. But first we will see the case $degP = 1$, up to $q - 1$-th root of unity, when δ and the class number is 1.

If $h = 1$, then as ψ is $\mathbf{F}_q$-linear,

$$g(q^j) = -\sum x^{-q^j}(x + x^q + \cdots + x^{q^{(f-1)}})\psi(1) = \Lambda_P = (-P)^{1/(q-1)} = \lambda$$

(This computation also shows why we defined Gauss sums by the character sums only for q^j's.) Now the rhs. is $-\lambda/\Pi_P(1/(1-q))$ which is λ, up to $q - 1$-th root of unity, by the results of chapter 2.

Lemma 3.11 *The Jacobi sum* $J_j = g(q^j)^q/g(q^{j+1})$ *is equal to* $-\underline{P}_{h-j}$, *where* $\underline{P}_a = \sigma^{1-a}\underline{P}$

$J_j^\sigma = J_{j+1}$ (j modhf) and J_{h-1} (we will prove below that it is an integer) is divisible by λ because of the proposition in the last section. Also by (4) of theorem 3.3, J_j is invariant wrt.μ-action and hence belongs to $K(\mu_{q^{hf}-1})$ and so is divisible by $\underline{P}_{h-j}$. But deg $J_j = 1$ by lemma 3.9, hence comparison of the degrees shows that J_j is $\zeta\underline{P}_{h-j}$, ζ being some $q^{hf} - 1$-th root of unity. We show that $\zeta = -1$, by looking at congruence at an infinite place. (I am grateful to prof.Tate for this suggestion) Fix z such that $z^{q-1} = -1/T$. As in lemma 3.6, but now at an infinite place, one can see that $K_\infty(\Lambda_P) = \mathbf{F}_q((z))$, for any infinite place of $K(\Lambda_P)$ above the infinite place of K. (For details, see Galovich and Rosen- Journal of number theory, 14, (1982), pa.159). Hence, $K(\mu_{q^{hf}-1})(\Lambda_P)_\infty = \mathbf{F}_{q^{hf}}((z))$. This field contains $g(1)$. Let σ denote the automorphism, which is q-th power on the constants and which fixes z, so that it fixes $K(\Lambda_P)_\infty$. By lemma 3.9,

$$g(1) = a_{-1}z^{-1} + a_0 + a_1 z + \cdots$$

where $a_i \in \mathbf{F}_{q^{hf}}$ and $a_{-1} \neq 0$. Hence,

$$g(1)^\sigma = -\zeta\frac{g(1)^q}{a^\sigma - T} = -\zeta z^{q-1}\frac{g(1)^q}{1 + a^\sigma z^{q-1}}$$

shows that $a_{-1}{}^q = -\zeta a_{-1}{}^q$, proving $\zeta = -1$ as desired.

Theorem 3.12 *(Analogue of the Stickelberger Theorem) For* $1 \leq k < q^{hf} - 1$, $g(k)/\lambda^{s(k)} \equiv 1/r(k) \bmod \overline{P}$ *where* $r(1) = 1, r(q^{j+1}) = -\underline{P}_{h-j}r(q^j)^q$, $r(\sum a_j q^j) = \prod r(q^j)^{a_j}, r(aq^h) = r(a)$ *and* $s(k) = k_0 + \cdots + k_{f-1}$ *where* $k = \sum k_i q^{hi}$ *is the digit expansion wrt.*q^h.

: For $k = 1$, this is proposition 3.7. Hence we assume that the theorem is true for $1 \leq k \leq t-1$ and try to prove it for $k = t$. Clearly, it is sufficient to consider $t = q^j \leq q^h$.

$$\frac{g(q^j)}{\lambda^{q^j}} = \frac{g(1)}{\lambda}\frac{g(q^j-1)}{\lambda^{q^j-1}}\frac{g(q^j)}{g(1)g(q^j-1)}$$

So it is sufficient to show that

$$\frac{g(q^j)}{g(1)g(q^j-1)}\frac{g(1)g(q^{j+1}-1)}{g(q^{j+1})} = \frac{g(q^j)^q}{g(q^{j+1})} \equiv \frac{r(q^{j+1})}{r(q^j)^q}\text{mod}\overline{P}$$

and this follows from the lemma.

It remains to prove the claim that $J_h = g(1)^q/g(1)^\sigma$ is an integer. Write $(g(1)) = (\overline{P}^{\sum r_i\sigma^{-i}})$. We have seen that r_i's are nonnegative integers and $r_0 = 1$. Since $J_h = \overline{P}^{(q-\sigma)\sum}$ belongs to F, $qr_i - r_{i+1}$ are divisible by $q^h - 1$. So $r_i = q^i\text{mod}q^h - 1$ for $0 \leq i \leq h-1$. But the theorem 3.8 shows that $\sum r_i = (q^h-1)/(q-1)$. Hence we can remove 'mod$q^h - 1$' from the equation. This reproves part of the full theorem and proves the claim, since $qr_i - r_{i+1} = 0$ or $q^h - 1$.

Hence the factorisation of both sides of the theorem 3.10 is the same and hence they are equal up to a $q^h - 1$-th root of unity. Comparison of both sides modulo $\overline{P}$ now proves the theorem as follows.

Reducing to the basic sums, it boils down to prove

$$g(q^j)/\lambda^{q^j} \equiv \frac{-1}{\Pi_P(q^j/(1-q^h))} \equiv 1/D_j\text{mod}\overline{P}$$

Now the Stickelberger theorem shows that the left hand side is congruent to the reciprocal of $(-\underline{P}_{h-j+1})(-\underline{P}_{h-j+2})^q\cdots(-\underline{P}_h)^{q^{j-1}}$. Since T is congruent to a, $-\underline{P}_k$ is congruent to $-a + a^{q^{h-k+1}} \equiv [h-k+1]$ and hence the desired congruence and the theorem follows. (Observe that $D_j = \Pi(q^j)$ means that $r(q^j)$ is an exact analogue of its classical counterpart)

We end this section by observing that comparison with the Morita's p-adic gamma function shows that the analogues are 'off by -1'.

3.4 Theorems of Weil and Deligne

Definition: Given (K, ∞) and a finite extension F of K, a group homomorphism χ from the group of fractional ideals (of the integral closure of A

in F) prime to ideal m to $\Omega^\times$ is called a Hecke character of F of conductor m, if for principal ideals (α) with $\alpha \equiv 1 \bmod m$,

$$\chi((\alpha)) = (\alpha/\mathrm{sgn}(\alpha))^\theta b^{\mathrm{ord}_\infty(\alpha)}$$

where $b \in \Omega$ and $\theta = \sum a_\sigma \sigma$ where the sum runs through K-embeddings σ of F into Ω and a_σ are integers. θ is then called the algebraic part of χ.

Our first goal in this section is to prove that the Jacobi sums (i.e.$n(\underline{a})$ is an integer) $g(\underline{a}, \wp)$ (as functions of $\wp$), extended multiplicatively, are Hecke characters of F.

Theorem 3.13 *(Analogue of Weil's theorem) If $n(\underline{a})$ is an integer, then the Jacobi sums $g(\underline{a}, \wp)$, extended multiplicatively, gives a Hecke character χ of F of conductor 1 with the algebraic part*

$$\theta = \sum n(\underline{a}^{(q^j)})\sigma^{-j}$$

where the sum runs over j in the Galois group $\mathbf{Z}/(r)$.

We have $g(1) = \eta_1 \eta_2{}^q \cdots \eta_{hf}{}^{q^{hf-1}}$ for some choice $\eta_i = (-\underline{P}_i)^{1/(q^{hf}-1)}$ (i modulo hf). Then lemma 3.11 shows that $g(q^j) = \eta_{1-j}\eta_{2-j}{}^q \cdots \eta_{hf-j}{}^{q^{hf}-1}$. Put

$$\beta_i = (\eta_i \eta_{i+r} \cdots \eta_{i+hf-r})^{(q^{hf}-1)/(q^r-1)}$$

(i modulo r). Then $g(q^j/(q^r-1), \wp) = \beta_{1-j}\beta_{2-j}{}^q \cdots \beta_{r-j}{}^{q^{r-1}}$. The hypothesis implies that there are integers t_i (i from 0 to $r-1$) such that $\sum t_i q^i = n(\underline{a})(q^r-1)$ and

$$\begin{aligned}
\chi((\wp)) &= \prod g(q^j/(q^r-1), \wp)^{t_j} \\
&= (\beta_1 \beta_2{}^q \cdots \beta_r{}^{q^{r-1}})^{t_0} (\beta_r \cdots)^{t_1} \cdots (\beta_2 \cdots \beta_1{}^{q^{r-1}})^{t_{r-1}} \\
&= \beta_1{}^{n(\underline{a})(q^r-1)} \beta_2{}^{n(\underline{a}^{(q)})(q^r-1)} \cdots \\
&= (-1)^{hf(n(\underline{a})+\cdots+n(\underline{a}^{(q^{r-1})}))/r} \wp_1{}^{n(\underline{a})} \cdots \wp_r{}^{n(\underline{a}^{(q^{r-1})})}
\end{aligned}$$

This proves the theorem with $b = (-1)^{\Sigma}$ where $\Sigma = \sum_{i=0}^{r-1} n(\underline{a}^{(q^i)})$.

From the theorem, we see that if $\underline{a}$ satisfies

$$(^{**})\ n(\underline{a}^{(q^j)}) \text{ is an integer independent of } j$$

then $g(\underline{a},\wp)/(\mathrm{N}\wp^{n(\underline{a})})$ is a Hecke character $\chi_{\underline{a}}(\wp)$ for F of finite order, and the theorem 3.10 shows that

$$\chi_{\underline{a}}(\wp) = (-1)^{n(\underline{a})f}/(\prod_{0}^{f-1} \Pi_P(-\underline{a}^{(q^{hi})})$$

as $(-1)^{n(\underline{a})(q^{hf}-1)} = 1$. Here, by abuse of notation, we put

$$\Pi_P(-\underline{a}) = \prod \Pi_P(-\langle a_i \rangle)^{m_i}$$

where $\underline{a} = \sum m_i a_i$. We go back to the formal situation of section 2.2 and adopting a similar convention.

Theorem 3.14 *If (**) holds, then*

$$f(-\underline{a}) = f(-1)^{n(\underline{a})}$$

Wlg. $N = q^r - 1$. Put $\underline{a} = \sum m_i a_i$,

$$Na_i = b_i = b_{r-1,i}q^{r-1} + b_{r-2,i}q^{r-2} + \cdots + b_{0,i}$$

For $1 \leq j \leq r-1$, (**) is equivalent to

$$(\sum m_i b_i)/N = (\sum m_i b_i q^j)/N - \sum m_i b_{r-1,i} q^{j-1} - \cdots - \sum m_i b_{r-j,i}$$

So

$$f(-\underline{a}) = \prod f(\frac{-q^j}{N})^{\sum m_i b_{j,i}} = f(\frac{1}{1-q})^{\sum m_i b_{r-1,i}} \prod f(\frac{-q^j}{N})^{\sum m_i (b_{j,i} - b_{r-1,i})}$$

Claim : $\sum m_i(b_{j,i} - b_{r-1,i}) = 0$. This is obvious if $j = r-1$, suppose it is true for $j = r-1, \cdots, r-t+1$. Then (**) for $j = t$ says that

$$\frac{\sum m_i b_i (q^t - 1)}{N} = [(\sum m_i b_{r-1,i})(q^{t-1} + q^{t-2} + \cdots + 1)] - [\sum m_i b_{r-1,i} + \sum m_i b_{r-t,i}]$$

Now (**) for $j = 1$ gives $(\sum m_i b_i (q-1))/N = \sum m_i b_{r-1,i}$ and hence the left hand side is equal to the first $[\cdots]$ and hence the second $[\cdots]$ is zero and the proof is complete.

Now we have seen in the last chapter that $M = \Gamma(0)/\tilde{\pi} = (-1)^{1/(q-1)}$, if $F_\wp$ denotes the Frobenius (q^{hf}-th power), then $F_\wp M/M = (-1)^{hf} = (-1)^{r \deg \wp}$.

Theorem 3.15 *(Analogue of Deligne's theorem) If (**) holds, and if we put* $E = \Pi(-\underline{a})/\tilde{\pi}^{n(\underline{a})}$, *then* $E^{\tau}/E = \chi_{\underline{a}}(\tau)$ *for any* $\tau \in \mathrm{Gal}(F^{sep}/F)$.

It is sufficient to look at action of $\tau = F_{\wp}$'s. But then by the preceeding theorem and the remarks just before and after it, we see that both sides are $(-1)^{hfn(\underline{a})}$. Hence the theorem follows. Another way to prove this theorem is to notice that $\chi_{\underline{a}}(\wp) = (-1)^{hfn(\underline{a})}$ by the proof of theorem 3.13.

Remark: As the gamma function in the function field case satisfies 'more' relations, we can handle more $\underline{a}$'s and at the same time, we can prove the full result and not just up to sign. We also do not need to use Kubert or Koblitz-Ogus results. It should be noted that the multiplication and the reflection formulae follow easily from the theorem 3.14. Also, in those cases, we can use $\Gamma(\underline{a})$ instead of $\Pi(-\underline{a})$ in the definition of E.

Let ρ be rank one $\mathbf{F}_q[T]$-module, given by $\rho_T = T + F$. Then $e_\rho(z) = \sum z^{q^j}/D_j$. But $D_j = \Pi(q^j)$. This should be compared with $e^z = \sum z^n/\Pi(n)$. We can rephrase this in more striking fashion as follows.

Classically, we have Hankel's formula

$$\frac{1}{\Pi(n)} = \frac{1}{2\pi i}\int_C e^z z^{-n}\frac{dz}{z}$$

which for a positive integer n, is visible residue (at $z = 0$) by the taylor series expansion; and from that point of view the same holds (with the classical exponential replaced by e_ρ) in the function field case, but now only for $n = q^j$. (we get 0 otherwise) The general value is then obtained multiplicatively. This should be compared with the situation for the Gauss sums in classical and the function field case. Gauss sums and factorials, both are 'sums' of a multiplicative character against a additive one. (Additional formal manipulation with $2\pi i$ replaced by $\tilde{\pi}$ and using reflection formula makes Hankel's formula 'look like' Euler's formula)

This analogy also suggests trying to make gamma functions, out of the coefficients of e_ρ, for other rank one Drinfeld modules ρ. The few cases I have tried we get gamma functions defined on $\mathbf{Z}_p$, but these are different than those defined in chapter 2.

We end this chapter by remarking that most of the results in this chapter are true in more generality than stated; but as we have not succeded in proving them in full generality yet, for simplicity we have used the restrictive hypothesis for which the proofs are most simple.

Chapter 4

Special values of Carlitz' zeta function and transcendence

This chapter can be read independently of chapters two and three. In this chapter, algebraicity questions for the values of the Carlitz zeta function at positive integers and for their ratios with the appropriate powers of the period are studied. The first section describes a formula for this values, gives two irrationality proofs and shows that some results of Wade [Wa 2] immediately give the transcendence in some cases. The second section deals with the ratios.

4.1 Values of the zeta function at positive integers

We saw in section 1.1 that the Carlitz zeta function is a zeta function with Euler product, v-adic interpolations; whose definition and special values at negative or 'even' positive integers are exact analogues of those for the Riemann zeta function. But the 'functional equation' is not known. This needs better understanding of the nature of zeta values. This, together with our poor knowledge of Riemann's zeta values at odd positive integers, leads us to consider the nature (by which I mean transcendence or the degree of algebraicity) of $\zeta(r)$ and $\zeta(r)/\tilde{\pi}^r$ for odd r (i.e. r not divisible by $q-1$).

A first remark one should make is that, for $q = 2$, there are no 'odd' values and we know by results of Carlitz-Wade (theorem 1.1) that in this case $\zeta(r)$ is transcendental and $\zeta(r)/\tilde{\pi}^r$ is rational for all positive integers r. If $q \neq 2$, and r is 'odd' then $\zeta(r)/\tilde{\pi}^r$ is not rational, since $\zeta(r)$ is in $\mathbf{F}_q((1/T))$ but $\tilde{\pi}^r$ is not.

Next, as the characteristic is p, the definition of the zeta function shows that $\zeta(rp^t) = (\zeta(r))^{p^t}$. Hence $\zeta(s)$ or $\zeta(s)/\tilde{\pi}^s$ is transcendental for $s = r$ iff it is so for $s = rp^t$. In fact, it is easy to see that, if F is a field and x is purely transcendental over F and $f(x) \in F((1/x))$ then $f(x)$ is algebraic of degree d over $F(x)$ iff $f(x^r)$ is algebraic of degree d over $F(x)$. Combining with the fact that $f(x^{p^t}) = f(x)^{p^t}$ (since the coefficients are in $\mathbf{F}_p$ rather than just in $\mathbf{F}_q$) in our case, we see that the nature of $\zeta(s)$ and $\zeta(s)/\tilde{\pi}^s$ is the same for $s = r$ as for $s = rp^t$, so that the results for one value will imply them for infinitely many values.

The first step is to get a more manageable formula for $\zeta(r)$. Put

$$[i] = T^{q^i} - T, L_i = [i]L_{i-1}, D_i = [i]D_{i-1}{}^q$$

$$[0] = L_0 = D_0 = 1$$

Now, the least common multiple of the monic polynomials of degree k is L_k. So $\sum_{\deg n = k} 1/n^r = a_{qkr}/L_k{}^r$ for some polynomial a_{qkr}. Experimenting with low values of q, k, r, I found that

$$\zeta(r) = \sum_{k=0}^{\infty} \frac{(-1)^{kr}}{L_k{}^r}, 1 \leq r \leq q \qquad (*)$$

which then I could prove by laborious computations involving symmetric functions. For $r > q$, a_{qkr} can be nonconstant polynomials. Later, I found essentially a formula for a_{qkr}, given without any mention of zeta, in Carlitz' paper ([Ca 1], thm. 9.4), which implies (*) and the general result. Unfortunately, it is given wrongly, but is easily correctable by slight straightforward changes. The argument involving 'generating functions' is much better than mine. The correct final result is : put

$$(1 - \sum_{i=0}^{k} \frac{(-1)^{k-i} t^{q^i}}{D_i L_{k-i}{}^{q^i}})^{-1} = \sum_{m=0}^{\infty} \Lambda_m^{(k)} t^m$$

then

$$\sum_{\deg n = k, n \text{monic}} \frac{1}{n^m} = \frac{(-1)^k \Lambda_m^{(k)}}{L_k} \qquad (**)$$

eg. for $1 \leq k$,

$$a_{qk(q+1)} = -[k]^q/[1] + 1, a_{qk(q+2)} = 2(-1)^{k-1}[k]^q/[1] + (-1)^{kq}$$

$$a_{qk(2q+1)} = -[k]^{2q}/[1]^2 + (-1)^k + (-1)^{k+1}[k]/[1]$$

Observe that in characteristic 2, $a_{qk(q+2)} = (-1)^{k(q+2)}$, as it should be, since we can divide by 2 and apply (*).

Now, consider r such that $\zeta(r) = \sum \pm 1/L_k{}^r$, eg. $r = mp^t, 1 \leq m \leq q$. Imitating the proof of the irrationality of e, we can give an extremely simple proof of

Claim : $\zeta(r)$ is irrational.

For if it is rational with denominator of degree m say, then $L_m\zeta(r)$ is a polynomial in T. Hence

$$[m]^r \cdots [1]^r \zeta(r) - (\frac{[m]^r \cdots [1]^r}{1} \pm \cdots \pm \frac{[m]^r \cdots [1]^r}{[m]^r \cdots [1]^r})$$

$$= \pm \frac{1}{[m+1]^r} \pm \frac{1}{[m+1]^r[m+2]^r} \pm \cdots$$

which is a contradiction since the left hand side of the equation is a polynomial, whereas the rhs is not (since it has the degree of the first term). This proves the claim.

Since the proof only involves degree and divisibility considerations, it works for say $r = (q+1)p^t, (q+2)p^t, (2q+1)p^t$ by (**).

Let us exploit this little more. (**) and the method shows that $\zeta(r)$ is irrational, if

$$\deg(\frac{(-1)^k \Lambda_{r-1}^{(k)} L_{k-1}{}^r}{L_k}) < 0$$

for sufficiently large k. i.e. if,

$$-\deg \Lambda_{r-1}^{(k)} > (r-1)(q^{k-1} + \cdots + q) - q^k$$

Put $f(i) = (-1)^{k-i}/(D_i L_{k-i}{}^{q^i})$ so that

$$\deg f(i) = -(iq^i + (q^k + \cdots + q^{i-1})) \leq \deg f(0)$$

and $\Lambda_{r-1}^{(k)}$ is the coefficient of t^{r-1} in

$$1 + (\sum_{k=0}^{k} f(i) t^{q^i}) + (\cdots)^2 + \cdots$$

Contributing terms are all of the form $\prod_{j=0}^{t}(f(j)t^{q^j})^{l_j}$ where $\sum l_j q^j = r-1$ and where t is least such that $q^{t+1} > r-1$.

$$-\deg \Lambda_{r-1}^{(k)} \geq -\max \sum_j \deg f(j) \geq -(\min \sum l_j) \deg f(0) \geq (\sum a_j)(q^k + \cdots + q)$$

where $r-1 = \sum a_j q^j$ is the digit expansion. So $\zeta(r)$ is irrational if $(1 + \sum a_j) \geq (r-1-\sum a_j)/(q-1)$. This condition is equivalent to $1+\sum a_j \geq r/q$ and an easy computation shows that it holds if and only if $r \leq q^2$ or $r-1 = q^2 + a_1 q + (q-1)$ with $0 \leq a_1 \leq q-1$.

On the other hand,

$$-\deg \Lambda_{r-1}^{(k)} = \sum l_j (j q^j + (q^k + \cdots + q^{j+1}) \leq (t+1)((r-1) + (q^k + \cdots + q))$$

since there are at most $t+1$ j's. Hence, the method fails if $\frac{r-1}{t+1}$ (or $\frac{r-1}{\log_q(r-1)}$) is large. I have not tried to explore the possibilities more systematically. In short, we have proved

Theorem 4.1 *$\zeta(r)$ is irrational, if $r = mp^t$, with $m \leq q^2$ or $m-1 = q^2 + a_1 q + (q-1)$ where $0 \leq a_1 \leq q-1$.*

Now we can prove the claim in another way, which 'suggests' transcendence and gives bounds for the degree of irrationality.

$$\deg(\zeta(r) - \sum_{k=0}^{m} \frac{\pm 1}{L_k{}^r}) = \deg(\sum_{m+1}^{\infty} \frac{\pm 1}{L_k{}^r}) = \deg(\frac{1}{L_{m+1}{}^r})$$

$\sum_0^m \pm 1/L_k{}^r$ is a polynomial divided by $L_m{}^r$. $\deg L_m{}^r = r(q^m + \cdots + q)$.

$$\frac{\deg L_{m+1}{}^r}{\deg L_m{}^r} = \frac{r(q^{m+1} + \cdots + q)}{r(q^m + \cdots + q)} = \frac{q^{m+1}-1}{q^m - 1} \to q$$

as $m \to \infty$. So by the analogue of the Liouville's theorem on approximation of algebraic numbers by rationals (this holds by essentially the same proof) we see that $\zeta(r)$ is transcendental or irrational of algebraic degree at least q.

So for $q > 2$, Roth's inequalities are satisfied, which, if we were in characteristic zero, would have implied transcendence. But straight analogue of the Roth's theorem fails in characteristic p, as was shown by Mahler by giving the following simple example. $a = \sum_{m=0}^{\infty} T^{-p^m}$ satisfies $a^p - a + \frac{1}{T} = 0$, so a is algebraic of degree p over $\mathbf{F}_q(T)$. But with $p_k = T^{p^k} \sum_0^k T^{-p^m}$ and $q_k = T^{p^k}$ we have $\deg(a - p_k/q_k) = -p \deg(q_k)$, so, in fact, Liouville's theorem can not be improved for this a. It will be interesting to know a good criterion saying when Roth's bounds would imply transcendence.

Here I should note that the claim made in Armitage's paper (Journal of Algebra 1968) that 'If a is not of p- power degree, then approximation by 'exponent' better than $2 + \epsilon$ implies a is transcendental.' is false, as was shown by Osgood (Indag.Math, 1975). Osgood, in fact, shows that 'If $(m, p) = 1$ or $m = p^t$ then there exists an a of degree m over $\mathbf{F}_q(T)$ for which the Liouville bound is the best possible. eg. $a = (1 + T^{-1})^{2/3}$ if $p > 3$.

But, fortunately in this case transcendence of $\zeta(r)$ can be proved anyway. Wade [Wa 1] has introduced a method of proving transcendence, by which he proves the transcendence of the Carlitz' period, which applies in this case. In fact, in [Wa 2] he proves that $\sum 1/L_k{}^r$ is transcendental and applies it to give another proof of transcendence of the Carlitz' period. Now, notice that from this transcendence of $\zeta(r)$ follows if $r = mp^t, 1 \leq m \leq q, p = 2$, or r is even. I have checked that the proof also works with $\sum(-1)^{kr}/L_k{}^r$ and it might even work in more generality. In any case we have,

Theorem 4.2 *$\zeta(r)$ is transcendental if $r = mp^t, 1 \leq m \leq q$*

4.2 The ratio $\frac{\zeta(r)}{\tilde{\pi}^r}$

If $q > 2$, $\tilde{\pi}$ does not belong to $\mathbf{F}_q((1/T))$, so put

$$\pi = \tilde{\pi}[1]^{-1/(q-1)} = \lim \frac{[1]^{\frac{q^n-1}{q-1}}}{L_n}$$

Then $\pi \in \mathbf{F}_q((1/T))$, and $\zeta(r)/\tilde{\pi}^r$ is transcendental iff $A_r = \zeta(r)/\pi^r$ is transcendental.

Let $r = mp^t, 1 \leq m \leq q - 2, (m, p) = 1$. From (*) in section one,

$$\zeta(r) = \lim \sum_{k=0}^{n} \frac{(-1)^{kr}}{L_k{}^r} = \lim \frac{c_n}{L_n{}^r}$$

with

$$c_n = [1]^r \cdots [n]^r + (-1)^r [2]^r \cdots [n]^r + \cdots + (-1)^{(n-1)r}[n]^r + (-1)^{nr}$$

i.e. $c_0 = 1$ and $c_{n+1} = [n+1]^r c_n + (-1)^{nr}$

$$A_r = \lim \frac{c_n}{[1]^{r(q^{n-1}+\cdots+1)}}$$

$$= [\frac{c_1}{[1]^r} + (\frac{c_2}{[1]^{r(q+1)}} - \frac{c_1}{[1]^r})$$

$$+ \cdots + (\frac{c_{n+1}}{[1]^{r(q^n+\cdots+1)}} - \frac{c_n}{[1]^{r(q^{n-1}+\cdots+1)}})] + r_n$$

where r_n is the remainder term. Put $[\cdots] = \frac{p_n}{q_n}$. Then

$$\deg(A_r - \frac{p_n}{q_n}) = \deg r_n$$

$$= \deg(c_{n+2} - [1]^{rq^{n+1}} c_{n+1}) - rq(q^{n+1} + \cdots + 1)$$

$$\begin{aligned} c_{n+2} - [1]^{rq^{n+1}} c_{n+1} &= ([n+2]^m - [1]^{mq^{n+1}})c_{n+1} \pm 1 \\ &= ((T^{q^{n+2}} - T)^m - (T^{q^{n+2}} - T^{q^{n+1}})^m)^{p^t} c_{n+1} \pm 1 \\ &= (mT^{(m-1)q^{n+2}+q^{n+1}} + \cdots)^{p^t} c_{n+1} \pm 1 \end{aligned}$$

Hence,

$$\begin{aligned} \deg r_n &= (r - p^t)q^{n+2} + p^t q^{n+1} + r(q^{n+1} + \cdots + q) - rq(q^{n+1} + \cdots + 1) \\ &= -p^t(q-1)q^{n+1} \end{aligned}$$

whereas $\deg q_n = rq(q^n + \cdots + 1)$. So,

$$-\frac{\deg r_n}{\deg q_n} \to \frac{(q-1)^2}{mq} = \frac{q-2}{m} + \frac{1}{mq}$$

Hence, Liouville's theorem implies

Theorem 4.3 *Let $r = mp^t, 1 \leq m \leq q-2, (m,p) = 1$. Then $\zeta(r)/\pi^r$ can be approximated to exponent at least $(q-1)^2/(mq)$. In particular, it is transcendental or irrational of algebraic degree at least $[\frac{q-2}{m}] + 1$.*

If $m \leq \frac{q-2}{2}$, then Roth's bounds are satisfied, but we are unable to conclude transcendence as explained in section one. Also, the theorem says nothing about $q = 2$ case, but we know it already.

Even though this 'suggests' transcendence, since we have not been able to prove it; the question arises as to what happens when we replace π with $\tilde{\pi}$. It is natural to consider $(\zeta(r)/\tilde{\pi}^r)^{q-1}$. Note that $\tilde{\pi}^{q-1} \in \mathbf{F}_q((1/T))$.

Proposition 4.4 *For $q = 3, r = 3^i$, $(\zeta(r)/\tilde{\pi}^r)^{q-1}$ is irrational.*

By earlier remarks it is enough to prove this for $r = 1$. Now, $(\zeta(1)/\tilde{\pi})^2 = \lim {c_n}^2/[1]^{3^n}$, where c_n is as before (for $r = 1$). Suppose it is rational. Then the denominator divides $[k-m]^{3^m}$ for some $k > m \geq 0$, because any irreducible polynomial of degree i divides $[i]$. In other words,

$$\lim \frac{(x^{3^k} - x^{3^m}){c_n}^2}{[1]^{3^n}} = \lim \frac{(x^{3^k} - x^{3^m}){c_n}^2}{x^{3^{n+1}}}$$

is a polynomial, so that for sufficiently large n, in the developement of $(x^{3^k} - x^{3^m}){c_n}^2$ in descending powers of x, there is a big gap (which tends to ∞ with n) after the power $x^{3^{n+1}}$ (which itself may not occur). But, we will obtain a contradiction by showing that $(3^{n+1} - 3^k)$-th power occurs (i.e. with non zero coefficient). Noting that $3^{n+1} - 3^k = 2(3^k + \cdots + 3^n)$, it remains to prove the

Claim : In the expansion of ${c_n}^2$, (A)$(3^k + 2(3^{k+1} + \cdots + 3^n))$-th power occurs but (B) $(\cdots) + 3^k - 3^m$-th does not (for any $k > m \geq 0$)

By the formula for c_n (and as coefficients are all ± 1 and as $2 \neq 0$), the terms of ${c_n}^2$ are of the form $\pm[l] \cdots [j-1][j]^2 \cdots [n]^2$ (empty product means 1).

Proof of (A): The power mentioned occurs as the highest power in $[k][k+1]^2 \cdots [n]^2$. We want to show that this is the only way it occurs and hence is not cancelled. In any term, where it occurs, it has the highest power contribution from $[k+1]^2 \cdots [n]^2$, otherwise the maximum power the term can contribute is $2(3 + \cdots + 3^k) + (3^k + 1) + 2(3^{k+2} + \cdots + 3^n)$ which is less than required. So we are reduced to showing that 3^k-th power can not be picked up from the first k levels. ($[i]$ or $[i]^2$ correspond to i-th level). $[k]^2$ can not occur in the term, for $2 * 3^k$ or $3^k + 1$ are too large and 2 is too small (as $2(3 + \cdots + 3^{k-1}) + 2 < 3^k$). But, if $[k]$ occurs, again the first power being too small as before, 3^k-th power is the only choice. (A) follows.

Proof of (B): For exactly the same reason as before, in any term containing the required power, the power has the highest power contribution from $[k+1]^2 \cdots [n]^2$. Hence, we want to show that we can not get $2*3^k - 3^m$-th power from the first k levels. Suppose we can. First, the case $m = 0$. Subcase I : k-th level is $[k]^2$: $2*3^k$ being too large and 2 being too small (just as before), x^{3^k+1} contributes. But then we need $3^k - 2$-th power from first $k-1$ levels, whereas the maximum you can get is $2(3+\cdots+3^{k-1}) = 3^k - 3$. So, this subcase is impossible. Subcase II: k-th level is $[k]$: Then x^{3^k} contributes and need $3^k - 1$-th power from the first $k-1$ levels, which we have seen to be impossible. Case of general m: Subcases as before. In subcase I, as before x^{3^k+1} is used and want $3^k - 3^m - 1$ from the first $k-1$ levels. Case $m = 0$ means that we can not get (from n levels) $3^{n+1} - 3^k - 1$-th power. Replacing n by $k-1$ and k by m,we see the impossibility of subcase I. In subcase II, again x^{3^k} is used. So we need $3^k - 3^m$ from the first $k-1$ levels, which now being [1],....,[$k-1$]; the maximum you get is $3+\cdots+3^{k-1} = (3^k-3)/2 \neq 3^k - 3^m$. Hence (B) is proved and the theorem follows.

Observe that the essence of the proof is in concluding the irrationality from the non-periodicity of the expansion. More results of this nature have been proved, but the irrationality of $(\zeta(r)/\tilde{\pi}^r)^{q-1}$ has not been done in general.

We end this chapter by observing that the zeta function for A or $\mathcal{A}$ can be defined, even when δ (the degree of the place infinity) is greater than one, by choosing instead of monic elements, n's with sign representatives from the chosen coset representatives of $\mathbf{F}_{q^\delta}^\times / \mathbf{F}_q^\times$, just as we have done for the gamma function. Then the outline indicated in chapter one, gives an expression for $\zeta((q-1)m)/\tilde{\pi}^{(q-1)m}$ in terms of the coefficients of the exponential for the corresponding Drinfeld module, so that its nature is tied up with the field of definition of the Drinfeld module.

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